1534

THE APPLICATION OF

GRAPHIC AND OTHER METHODS

TO THE

DESIGN OF STRUCTURES.

(SPECIALLY PREPARED FOR THE USE OF ENGINEERS.)

BY

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TO

JAMES BUSH, B.Sc. (Lond.),

AS A SLIGHT EXPRESSION

OF A KEEN APPRECIATION OF HIS UNTIRING AND EMINENTLY

SUCCESSFUL WORK AMONGST THE STUDENTS

OF THE

CARDIFF SCIENCE AND ART SCHOOL,

DURING THE TWENTY-FIVE YEARS OF HIS PRINCIPALSHIP,

THIS WORK IS AFFECTIONATELY DEDICATED

BY

ONE OF HIS OLD STUDENTS,

THE AUTHOR.

PREFACE TO FIRST EDITION.

In compiling this work, the Author has tried to make prominent the fact that Graphic Methods are only the instruments by which particular numerical results are often easily obtained, and if considered apart from physical conditions and quantities, they are simply mathematical exercises, and of little real use to the engineer or the engineering student. For this reason some of the designs have been worked out in detail, and the chapter on working stress has been written.

The sequence of chapters might have been altered perhaps with advantage; but as many of them are comparatively complete in themselves, this is not a matter of great importance. The reason for introducing the theory of bending so early was, that in the Author's opinion it is preferable to show the application to concrete structures of the various graphic methods given in the text; and as that involved the design of a strut, and that again involved the theory of bending, the Author did not hesitate to introduce it where he did. Should any reader not be familiar with the different methods of deducing results there shown, he may pass them over and use the results in the design he has in hand.

Much has been introduced into this work which is not to be found in many books on Graphic Statics, such as the theory of counterbracing, maximum bending moment with moving loads, design of a plate girder, stability of masonry structures, masonry and metal arches, and the theory of structures containing redundant members; but the Author has not attempted to treat these sections exhaustively from every point of view, but rather to introduce them in their general aspect, and thus pave the way for a further study of works much larger than this. At the same time it is hoped that the reader may be able to find in these pages much that will be useful to him in his ordinary work.

London, May, 1896. W. W. F. P.

PREFACE TO SECOND EDITION.

As a second edition of this work has been called for, the Author thought it expedient to re-write the chapter on Struts, and to deal with the matter more systematically than in the first edition.

He has also added some notes in the form of an Appendix where he considered that such may be of use in the further elucidation of points in the original text.

W. W. F. P.

London, 1905.

CONTENTS.

SYMBOLS GENERALLY USED IN THE TEXT.

f = stress intensity per square inch.
E = Young's modulus of elasticity.
F = applied force.
M = moment; generally bending moment.
h = distance from neutral axis to outside fibre in beams.
h = rise in arch.
H = horizontal component in arch ring.
I = geometrical moment of inertia about an axis through the centre of gravity of section.
R = radius of curvature in beams and bending.
R = resultant force in connection with masonry.
R = total stress in member produced by loads when redundant member is removed.
R = reaction at end of beam.
w = load.
l = length.
d = depth or thickness.
A = area.
ρ = radius of gyration.
n in struts given by equation $d = n\,\rho$. See page 78.
c in struts = constant. See pages 78, 88 and 99.
$k = \frac{A}{\rho^2}$ in struts. See table, pages 152 and 153.
$m = \frac{l}{A\,E}$ in equation $S\,m$ = elongation.
S = total stress in member, in problems requiring the principle of work for solution.
δl = elongation of l, $= S\,m$.

GRAPHIC STATICS.

CHAPTER I.

INTRODUCTORY.

THE term "graphic statics" is generally meant to imply the art of solving some useful statical problems graphically—that is, by the mere drawing of lines. Since the solution of problems other than statical is given by one or more equations, and equations can be represented graphically by curves, it is clear that the solution of problems of almost any description may be represented graphically, and in many cases problems may be completely solved by graphical methods; hence it is not surprising to find under the heading "Graphic Statics" problems solved graphically which are more nearly related to other branches of natural philosophy.

A diagram or curve in itself represents a mathematical fact or relationship altogether independent of any physical phenomena, and it is only when the several components of the figure are assumed to represent physical quantities that the diagram can bear any physical interpretation. A diagram may fulfil two different objects: it may present to the eye a picture representation of the intensity or degree of certain phenomena; or it may be used to obtain a solution to a problem by the mere drawing of lines. In both instances it may be found to possess manifold advantages over other processes of representation or solution, but in all cases judgment should be exercised in its application.

Its relation to engineering work is much of the same nature as that of the slide rule or mechanical calculator. It is a means to an end, whereby a more or less complicated mathematical operation is simplified and facilitated, the result being often obtained with the expenditure of but an infinitesimal amount of brain power. The study of "graphics" in the abstract only would be of little use, except as a factor of mathematics; it is in its application to the concrete problems of the engineer and the physicist that its intrinsic value becomes apparent. Even then it is only a portion of the whole problem to be solved; for let

us take as an example the design of a roof truss. The drawing of the stress diagram is only the beginning of the design. The compression members have to be designed to resist crippling as well as direct compression, while those members that resist external transverse forces have to be made strong enough to resist bending as well as the other straining actions. It is thus evident that graphic statics is intimately related to, and mixed up with, structural design in general, and it is on this account that some of the succeeding problems will be worked out in full, and not merely the stress diagram given. In this way it is hoped that some of these notes may be of more than passing interest to the draughtsman or designer, and that the student may form an idea of the ultimate object of the study of graphics.

Graphic Solution of Equations.

Although not of frequent occurrence, it is sometimes necessary to solve an equation containing a higher power of the unknown quantity than the second. It may by chance be solved algebraically in a few isolated cases, but it will be found that in most instances that occur in practice a solution may be easily obtained by a graphic method.

As a first example, let us take a quadratic equation, because the solution may be obtained by easy algebraical methods, thus forming a check upon the graphic solution. The algebraical solution of the equation

$$x^2 + 2x - 3 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

gives us $x = 1$ and -3. Draw two lines at right angles, such as XX^1 and YY^1, intersecting at O, the origin (fig. 1) Measure the several values of x from the line YY^1, parallel to XX^1, and the values of y parallel to YY^1 from the line XX^1. The positive values of x and y are measured to the right and upwards, while the negative values are measured to the left and downwards, respectively.

The lines XX^1 and YY^1 are called the axes of x and y respectively.

Now we may write equation (1) in the form

$$x^2 + 2x - 3 = y \quad . \quad . \quad . \quad . \quad . \quad (2)$$

equation (1) being really only a special form of (2), when y is made equal to zero. Give to x any values which may seem convenient, such as natural integers 1, 2, 3, &c., and we obtain a corresponding series of values for y. Again, by substituting for x negative integers, another series of values

of y will be found. Plot off these pairs of values of x and y as in fig 1. Thus, when $x = 1$, $y = o$; therefore measure off from O along O X, a distance O A, equal to unity. This is a point on the curve. Again, in equation (2) $x = 2$, $y = 5$. Hence set off O B = 2 and B C = 5; then C is a point on the curve. When $x = o$, $y = -3$; therefore measure off from O along O Y^1 in the negative direction O D = 3. Also,

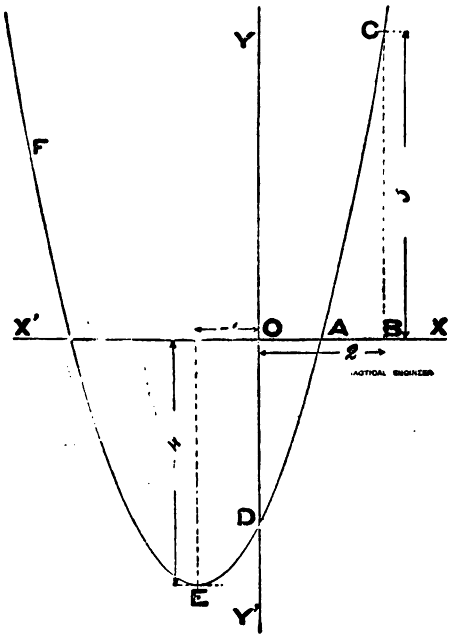

Fig 1.

when $x = -1$, $y = -4$, which, when plotted, locates the point E. If this be carried on with a few more values of x and y, and the points so obtained joined up in series, we obtain the parabolic curve F E D A C, which is the graphical interpretation of the above equation (2). The particular value of y which gives us equation (1) is zero; therefore what values of x correspond to $y = o$. Now, y is measured

[illegible] in [illegible] will [illegible] all points [illegible]. The [illegible], which [illegible] respectively.

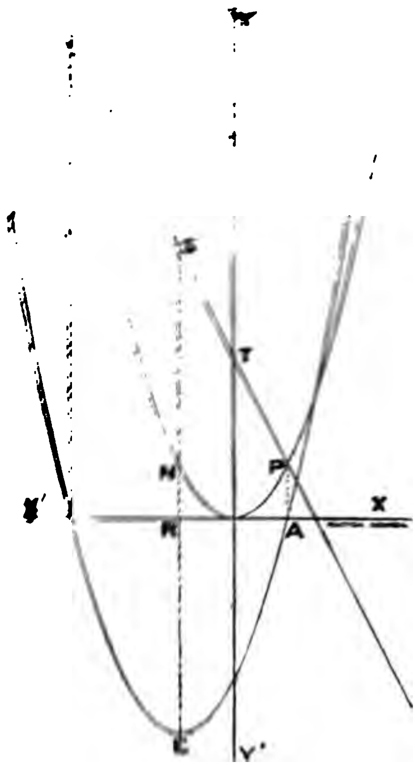

FIG. 2.

These numbers are the roots of equation (1), found algebraically. This method, applied to any equation containing only one variable or unknown quantity, will

provide a solution. Virtually it is only required to draw the curve accurately near the points A and A^1, where it cuts the axis of x; but, at the same time, it should be drawn roughly throughout its different convolutions, so as to obtain a fair idea of the form of the curve.

This process may be considerably simplified, as may be seen from what immediately follows. Equation (1) may be written thus—

$$x^2 = 3 - 2x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let $\qquad x^2 = y$, and $3 - 2x = y^1 \quad . \quad . \quad . \quad . \quad . \quad (4)$

also let $\qquad x^2 + 2x - 3 = Y \quad . \quad . \quad . \quad . \quad . \quad (5)$

then $\qquad Y = y - y^1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$

But equation (1) is a particular case of equation (5), when $Y = o$; and therefore, when $y - y^1 = o$, or $y = y^1$—that is, when an ordinate to the curve—$y = x^2$ (for a particular value of x) equals the ordinate to the curve $y^1 = 3 - 2x$ (for the same value of x). This can only happen where the two curves, $y = x^2$ and $y^1 = 3 - 2x$, intersect; hence, plot the two curves represented by equation (4). These will be the parabola M O Q, and the straight line M P, fig. 2; they intersect in M and P. The numerical equivalents of the abscissæ of these points—namely, of O A^1 and O A—are the roots of the equation—

$$x^2 + 2x - 3 = 0.$$

This method of solution is now obvious. Draw the curve represented by the equation $x^2 = y$, and the straight line given by $3 - 2x = y$. The abscissæ of their points of intersection represent the roots of the equation.

If we analyse figs. 1 and 2, we shall find that the latter is only the former dissected; and we can easily compound the two curves of fig. 2, so as to produce that of fig. 1. Thus, from equations (5) and (6), we see that

$$y - y^1 = Y = x^2 + 2x - 3;$$

or, in other words, an ordinate Y, in fig. 1, is the same as the ordinate y^1 of the straight line, fig. 2, subtracted from the similarly-situated ordinate y of the parabola. In the figure, when

$$x = -1, y = 1 = RN,$$

and $\qquad y^1 = 5 = RS;$

and hence

$$Y = y - y^1 = RN - RS = RE = -4.$$

At P, where the line and parabola intersect,

$$Y = y - y^1 = AP - PA = 0;$$

and hence the corresponding value of x gives one of the roots of the equation.

This idea may be extended to equations other than quadratics. Take the equation—

$$x^3 - x^2 + 3x - 1 = \circ = y, \text{ say.}$$

This is made up of three separate equations, namely—

$$x^3 = y_1, \; x^2 = y_2, \text{ and } 3x - 1 = y_3.$$

When taken together, we have—

$$y = y_1 - y_2 + y_3$$

for the same value of x, and hence the sum of the first and third ordinates, minus the second, gives the value of the ordinate of the curve—

$$x^3 - x^2 + 3x - 1 = y.$$

If a solution of this equation is required, it will be found rather easier for the beginner to draw the two curves represented by—

$$x^3 = y, \text{ and } x^2 - 3x + 1 = y^1.$$

The abscissa of their intersection gives the real root of the equation, the other two being imaginary.

It will be found convenient to plot the curves $x^4 = y$, $x^3 = y$, and $x^2 = y$, on a piece of squared paper (fig. 3); the larger the paper the better. The squares should be divided into ten equal parts, and every tenth line should be thickened. This expedites the plotting considerably. For the sake of clearness in the figure, the squares have not been divided into ten equal parts, the figure being so small.

Now, should it be required to solve the equation—

$$x^3 - 4x - 2 = 0,$$

all that remains to be done is to draw the straight line

$$y = 4x + 2,$$

which can be done by merely plotting two points in the line. The curve $y = x^3$ is already drawn, and the intersections A, B, and C of this curve and the line give the solution of the above equation. It will be noticed that the intersection occurs at three points; hence there are three real roots to this equation. They are $x = 2{\cdot}2$, $x = -{\cdot}53$, and $x = -1{\cdot}64$. Instead of actually drawing the line $y = 4x + 2$, a very

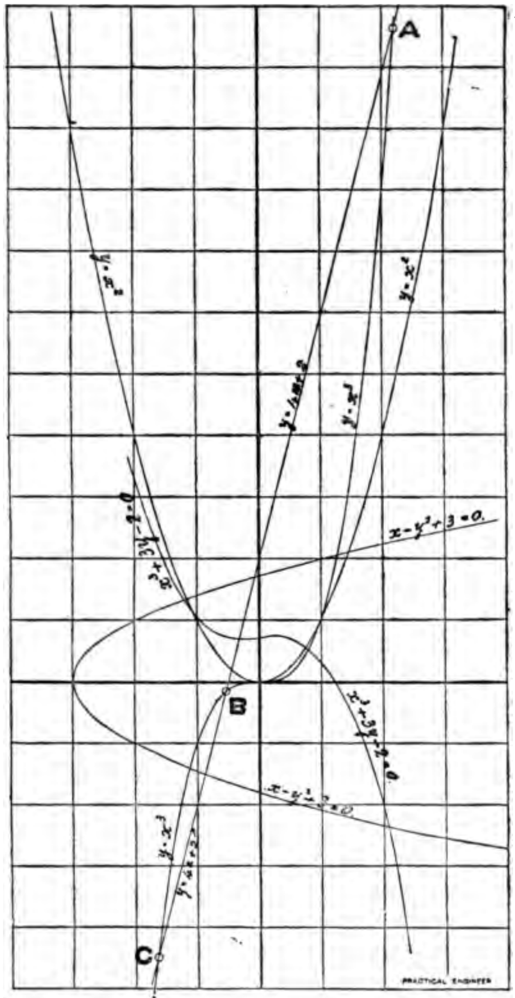

FIG. 3.

fine piece of wire or silk thread can be stretched between two points in it, and the points of intersection noted. In this way the squared paper does not become filled up with lines. This method is recommended by Prof. R. H. Smith in his treatise on "Graphics."

Simultaneous equations may be treated in a similar manner. Let it be required to solve the equation—

$$\left.\begin{aligned} x^3 + 3y - 2 = 0 \\ x - y^2 + 3 = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (7)$$

Plot both of these curves, and the co-ordinates of their points of intersection are the real roots of the above simultaneous equation (see fig. 3). They are $x = 2{\cdot}09$, $y = -2{\cdot}35$, and $x = -1{\cdot}2$, $y = 1{\cdot}32$. The equation—

$$3 \log_{10} x - 4x + 24 = 0 \quad \ldots\ldots \quad (8)$$

is easily solved by first plotting the curve—

$$y = \log_{10} x,$$

and then the straight line—

$$3y - 4x + 24 = 0.$$

The abscissæ of the intersections, as usual, give the roots.

Equations containing trigonometrical functions are just as easily solved.

It will be noticed in fig. 3 that the curves generally have a greater range in the vertical than in the horizontal direction. Should this be inconvenient, the vertical scale can be taken a fraction of the horizontal scale, say one-tenth or one-hundredth. This will have the effect of widening the curves out considerably.

Diagrams in Three Dimensions.

Up to the present time the only equations taken have been those in which there were only two variables, and they have in every case represented curves that existed wholly in one plane. The two variables represented the dimensions of the curve in the plane when measured from well-known and recognised fixed lines called axes. If now an equation should contain three variables instead of two, it is evident that it cannot be represented by a single curve in one plane, as heretofore, but will necessitate another dimension or direction in which to measure the third variable. In the previous cases, the two axes were taken at right angles to one another. Now add another axis

which is perpendicular to each of the other two, and which passes through the origin. This axis denotes the direction in which the third variable should be measured.

As it is not possible to represent a solid in the same way as a plane figure, we must resort to some sort of projection on a plane.

As an example, let us take the equation—

$$P V = R T . . . . (9)$$

which represents the relation between the pressure volume and temperature of a gas, in which P = pressure per square foot, V = volume in cubic feet, R = a constant = 53·2 for air, and T = the absolute temperature on the Fahrenheit scale. For the sake of plotting it is easier to use pounds per square

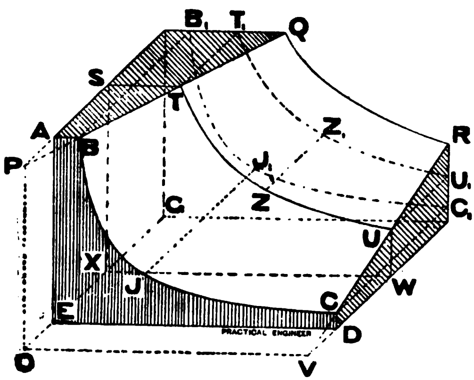

Fig. 4.

inch, which, if represented by p, the above equation becomes p V = ·368 T (for air). Now, from O, fig. 4, draw O P and O V at right angles, these being the axes of pressures and volumes respectively. Then take O G inclined at any angle, say 45 deg., to O V. It is in reality perpendicular to the plane containing O P and O V, and along this line or parallel to it the absolute temperatures are to be measured.

Give any value to T in the above equation, say 493, corresponding to 32 on the ordinary scale (assuming absolute zero to be at − 461 deg.), and measure off O E = 493. Then whatever values p and V can have in the equation p. V. = ·368 × 493 will be plotted on the plane through E parallel to the plane containing P O V, namely, A E D. This

plane is shaded with vertical hatchings for the purpose of emphasising it. The pressures are only taken up to 120 lb. per square inch, and the volume to 18·2 cubic feet. Hence A E = O P = 120 lb., and E D = OV = 18·2 cubic feet. The points in the curve B J C are obtained by giving values to V, and solving the equation for p. The values given to V are plotted off along E D from E, and then the corresponding values of p erected at the ends of those abscissæ.

In the same way the curve T Z U is plotted in the plane S X W, when the temperature is assumed to be 1,000 deg. Fah.—that is, 1,461 deg. on the absolute scale, and therefore O X = 1,461 = T. Again, the curve Q R is plotted for an absolute temperature of 2,461 deg.

If we join up the terminals B, T, Q of the curves, we find they all lie on a straight line through the point P. Similarly, R U C V is a straight line. This fact might have been anticipated from a glance at the original equation, which may be written as—

$$V = \frac{\cdot 368}{p} \cdot T.$$

All the points Q, T, B, and P are at the same height above the base of the diagram, namely, 120 lb. As p is here constant, the equation shows that V varies as the temperature T, the equation being of the first degree; Q P must necessarily be a straight line. The same reasoning also applies to R V. The hollow-curved surface Q B C R is the graphical representation of equation (9), in the same way that the plane curves in previous figures represent graphically the respective equations from which they were derived. For most purposes the equation can be converted temporarily into one of two dimensions, by giving one of the variables a definite value; the corresponding plane curve can then be plotted. If other values are in turn given to the same variable, and the several curves plotted, we shall have a series of curves of the same kind, apparently in the same plane, which, with certain limitations, will represent the curved surface to a certain extent. Three such curves are Q R, T_1 U_1, and B_1 C_1, all of which are in the plane F G H. But, upon a closer inspection, it is manifest that these curves are no more than the curves Q R, T U, and B C projected on the plane F G H by lines parallel to the axis O G.

This diagram belongs, strictly speaking, to thermodynamics, but it is here given as a fair representation of an equation containing three variables.

Drawing Parallel Lines.

The essence of graphic work in general depends upon the accuracy with which a line may be drawn parallel to another line on any portion of a sheet of paper with ease and celerity. An ordinary rolling parallel rule may be used, but to be of general use it must be of considerable length, at least of from 1½ ft. to 2 ft. The longer the rule the more nearly parallel the line, but the longer the rule the quicker one end moves off the drawing board, so that, at best, it is only a compromise. Two set squares may be used, but if the line to be drawn is a long way from its parallel companion, there is every chance of the final line being considerably out of parallelism, besides the inconvenience

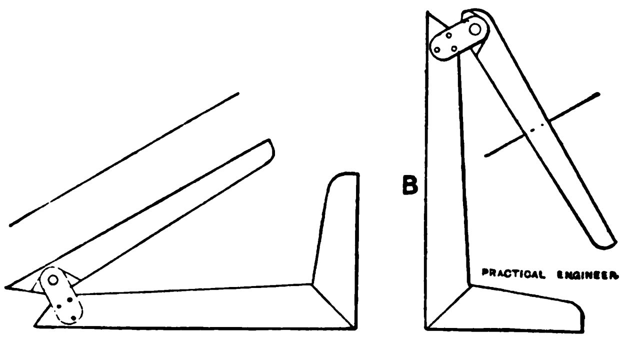

Fig. 5.

of shuffling the set squares one after another down the board.

The only way in which diagrams containing parallel lines can be quickly and accurately drawn is by the use of the clinograph, fig. 5, which was invented and patented by Mr. Joseph Harrison, Wh.Sc., A.M.I.C.E. It consists of a couple of arms connected by a stiff joint, one of which slides along the edge of the T square, the inclination of the other being adjusted so that its edge coincides with the line to which another has to be drawn parallel. By placing the foot against the edge of the T square, a perpendicular, instead of parallel, line can be drawn (see B, fig. 5). The author has found that work with this instrument may be still facilitated by cutting off the pointed end of the arm

near the joint, and shaping it in the form of a circle, to which the drawing edge is a tangent (see fig. 5, B). In this way the arm may be directly set parallel to a given line by bringing the circular end up to the line until it touches it, and then swinging the arm round until its drawing edge coincides with the given line. The circular end should have its centre in the axis of the rivet pin of the joint. This instrument will also be found of great use to students in practical, plane, and solid geometry.

CHAPTER II.

The Laws of Equilibrium.

The whole of the subject "Statics" is based upon the two laws of equilibrium, which fact has hardly been emphasised as it should have been in most of the student's text-books. Such abortive terms as the "principle of the lever," so often to be found in the applied mechanics examination papers of the Science and Art Department, would perhaps never be used at the present time had the principles of equilibrium been taught as they should have been.

A body may be said to be in equilibrium when all the forces acting on the body mutually balance each other, both as regards translational and rotational effect. It may, perhaps, be put in another way, thus: Let a body be *perfectly free to move*. Any force, either great or small, will, if applied to the body in the above state, make it move: the greater the force applied, the greater will be the velocity produced in the body. If now a body, which is perfectly free to move, does not move, then, clearly, there can be no resultant force acting upon it—that is, the forces acting mutually balance each other. This state of balanced forces is called *equilibrium*. The two laws of equilibrium may now be cited.

If any number of forces act upon a rigid body, and maintain it in equilibrium, then—

I. *The sum of the components of these forces in any one direction must be zero*; and

II. *The sum of the moments of these forces about any point in their plane must be zero.*

These two laws are almost self-evident; for, in the first, if the sum of the components in any one direction be not zero, it must be some definite resultant force, which will

make the body move. But as the body does not move, then there can be no resultant force in any direction. Similarly with the sum of the turning effects or moments of the forces. Because the body does not turn, there can be no resultant turning effort acting upon it—that is, the sum of the moments must be zero. These results will be more deeply impressed upon the student's mind if he will take a few cases of equilibrium and draw the diagram of forces to scale.

There are a couple of results in connection with the equilibrium of a body which it will be well to bear in mind.

The first is that *the force diagram of a body in equilibrium must be closed.* This will be seen at once, thus: If the polygon of forces (force diagram) of a body *not* in equilibrium be drawn, the polygon will not be complete, but will require

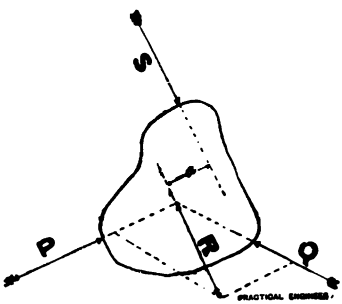

FIG. 6.

a single line to be drawn direct from the starting point to the finishing point, before it is completed or closed. This line represents the resultant of the set of forces. If, now, the body is in equilibrium, there can be no resultant force, and therefore no line will be required to be drawn to complete the polygon; or, in other words, the polygon will be complete without it.

Another is, that *if only three forces act upon a body in equilibrium, the directions of these forces must all pass through the same point.* For, if possible, let the three forces P, Q, and S maintain the body, fig. 6, in equilibrium. The resultant of P and Q must be R by the parallelogram of forces, and its direction must be parallel, equal, and opposite to S, from the first law of equilibrium. If the direction of S does not pass through the intersection A of P and Q, it

must lie at some distance a from it. Then the two equal and opposite forces R and S will produce a couple tending to rotate the body, equal to S a. But as the body is in equilibrium, there can be no resultant turning effort acting, and hence S a must be zero. As S cannot be zero, then a must be zero—that is, S must pass through A, the intersection of the other two forces.

Again, *if a hinged rod in equilibrium is acted upon by a number of forces at its extremities only, then the stress in the rod must act along its axis.*

In fig. 7 let one end of the rod A B be acted upon by the forces F_1, F_2, F_3, and F_4. These forces will have a resultant, say R_1. From the first law of equilibrium, there must be

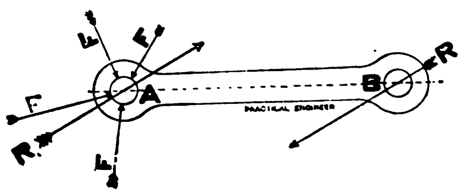

FIG. 7.

an equal parallel and opposite force R_2 at the other end Each of these can be resolved parallel and perpendicular to the axis of the rod. The parallel components will be equal and opposite, and the perpendicular components will form a couple, whose moment is the length of rod multiplied by the component. But as the rod is in equilibrium, there can be no resultant couple acting upon it; therefore the moment of this couple must be zero—that is, the perpendicular component must be zero; and hence the resultant force on each end must be wholly along the axis. The internal force, or stress, which resists the external force must be equal and opposite, and therefore must act along the axis of the rod.

CHAPTER III.

SHEARING FORCE AND BENDING MOMENT.

IF there is any excuse needed to account for the introduction of the investigation of shearing and bending action at this early stage, before considering the cases of simple

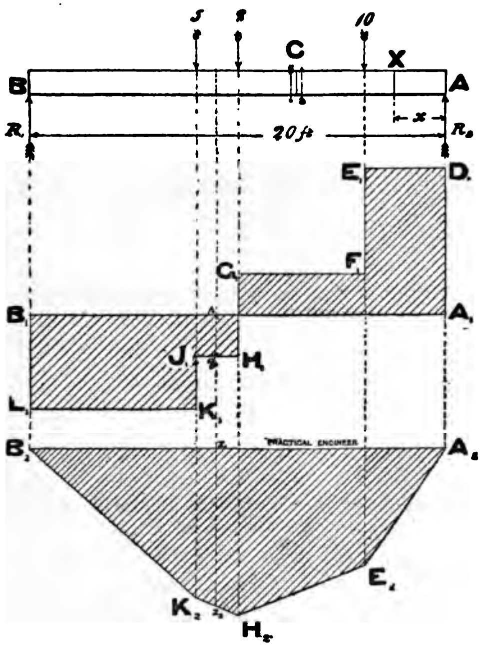

FIG. 8.

direct stress, it is that in so doing we meet with the graphical determinations of the conditions of equilibrium of a body, which will be more or less used throughout the whole of Graphic Statics.

The author has often found students to possess very hazy notions as to the meaning of the terms shearing force and bending moment, and he therefore proposes here to approach

the matter from a very elementary point of view, in the hope that, by so doing, some of the intellectual fog may be cleared away. The beam, fig. 8, is supported at both ends, and loaded with three weights of 5, 8, and 10 tons, as shown. The supporting forces are at once obtained from the application of the two laws of equilibrium. From the first we get the sum of the external forces acting on the beam is zero, or

$$R_1 + R_2 - 5 - 8 - 10 = 0 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

From the second law, the sum of the moments of all the forces about any point in the plane of the forces is zero.

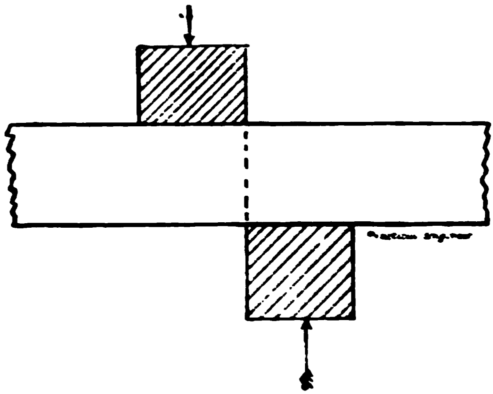

FIG. 9.

Take moments about any point A in the line of action of the right-hand supporting force, and we get

$$20\,R_1 - (5 \times 12) - (8 \times 10) - (10 \times 4) = 0 \quad . \quad . \quad (11)$$

therefore $R_1 = 9$ tons.

Substituting this value of R_1 in (10), we find that $R_2 = 14$ tons.

This operation of finding the supporting forces is nearly always the first to be carried out in finding the bending moments, shearing forces, or the stresses in a hinged structure.

SHEARING FORCE.

Take any transverse section of the beam, say that at C, and find the sum of all the forces acting on that part of the beam to the right of the section C.

$$\text{Sum of forces} = R_2 - 10 = 14 - 10 = 4 \text{ tons.}$$

In other words, the resultant of all the forces to the right of C is 4 tons ; or the forces to the right of C are tending to make the portion A C of the beam move bodily upwards in the same way that a single upward force of 4 tons would do.

Again, in the same manner, the sum of all the forces to the left of C = $R_1 - 5 - 8 = 9 - 13 = -4$ tons, the negative sign showing that this resultant force acts in the negative or downward direction.

Hence we have on each side of the section C a resultant force of 4 tons—one urging the right-hand portion of the beam in the upward direction, the other urging the left-hand portion in the downward direction. This action is precisely the same as would be produced if the beam were inserted between the knives of a hydraulic shearing machine, and the knives replaced by simple blocks of metal, as indicated by the shaded portions of fig. 9, and a total pressure of 4 tons brought to bear upon the hydraulic piston (neglecting friction). This action is called a shearing action, and either force of 4 tons is called a shearing force.

Obviously, then, *the shearing force over any transverse section of a loaded beam is the sum of the forces on one side of the section ;* either side being taken at will.

The shearing force at any section of a beam may be represented graphically by plotting ordinates at different points along the axis of the beam, the length of the ordinates being proportional to the shearing force. Thus, the shearing force near the end A, fig. 8, must be R_2 (neglecting the weight of the beam), that being the sum of the forces to the right of the section. This shearing force remains constant as the section is moved from A to the line of action of the load, 10 tons. Between the loads 10 tons and 8 tons the sum of the forces to the right of the section will be $R_2 - 10 = 4$ tons, and this is constant between these loads. Similarly, between the 8 tons and 5 tons the shearing force is $R_2 - 10 - 8 = -4$; and between the latter load and the end the shearing force is $R_2 - 10 - 8 - 5 = -9$ tons.

Draw a line $A_1 B_1$ parallel to the axis of the beam, and of the same length, and plot the ordinates representing the shearing forces. Beginning at A_1, assuming the upward direction as positive, we plot off $A_1 D_1 = R_2 = 14$ tons to some convenient scale. As the shearing force is constant between the loads, the upper ends of all the ordinates between D_1 and E_1 must lie in the line $E_1 D_1$. At f erect the ordinate $f F_1 = R_2 - 10 = 4$ tons. The line $G F_1$ will be horizontal for the same reason that $E_1 D_1$ was horizontal.

Proceeding in this way, between h and g, the shearing force was found to be -4 tons; therefore, plot $g\,H_1 = -4$—that is, equal to 4 tons in the downward or negative direction.

Completing the diagram, and drawing the cross lines for the sake of emphasis, we get a representation of the shearing force throughout the whole length of the beam. To prevent any confusion of signs, it is better to keep to one side of the section throughout the whole length of the beam when computing the shearing forces. There follows, from what has just been done, a simple mechanical rule for drawing the shearing force diagram. Begin at one end of the beam, and draw a zigzag line, such that the vertical portions represent in magnitude and direction the forces acting on the beam. Thus, at A_1 erect an ordinate $A_1\,D_1 = 14$, in the same direction as R_2. Draw the horizontal line $D_1\,E_1$ to the line of action of the next force. From E_1 draw $E_1\,F_1 = 10$, in the same direction as the force 10 tons; then, after completing the horizontal line $F_1\,G_1$, draw $G_1\,H_1$ parallel to the force 8 tons, and equal to it. In the same way, $J_1\,K_1 = 5$ tons, and $L_1\,B_1 = 9$ tons.

BENDING MOMENT.

Take any transverse section C, as before, at a distance of say 8 ft. from the point A. Now the sum of the moments of all the forces acting on the beam to the right of C about the axis of the section C is $8\,R_2 - (10 \times 4) = 72$ tons-feet in anti-watch-hand direction, which we may call positive. Again, the sum of the moments of all the forces to the left of C about the axis of the section $C = -(R_1 \times 12) + (5 \times 4) + (8 \times 2) = -72$ tons-feet. The two resultant moments about the section C are equal and opposite in sign or direction, and hence their sum must be zero, which we should naturally expect from the second law of equilibrium. Either of these moments is called the bending moment at the section C. Hence we define the bending moment at a section as *the sum of the moments of all the forces on one side of the section*. It will be noticed that all moments which tend to make the beam turn about the section C in watch-hand direction are called *negative* moments; and all those tending to make the beam turn in anti-watch-hand direction are called *positive*. In the same manner all forces acting in some particular direction are considered positive; those in the opposite direction negative. The particular direction may be chosen at will.

The bending moment at any section X between R_2 and the load 10 tons, distant x feet from $R_2 = R_2\,x = 14\,x$ tons-

feet $= y$, say. Then, when $x = 0$, $y = 0$; also when $x = 4$, $y = 56$; and as the equation $y = 14x$ is of the first degree in x, and y similar to the second part of equation (4) it therefore represents a straight line. Hence measure off from A_2, $x = 4$ in the left-hand direction, and erect the ordinate $y = 56$, giving the point E_2. Join $A_2 E_2$. Again, take another section C between the loads 10 and 8 at a distance x ft. from A. The bending moment at $C =$ sum of moments of forces to right of $C = (R_2 \times x) - 10(x - 4) = 4x$

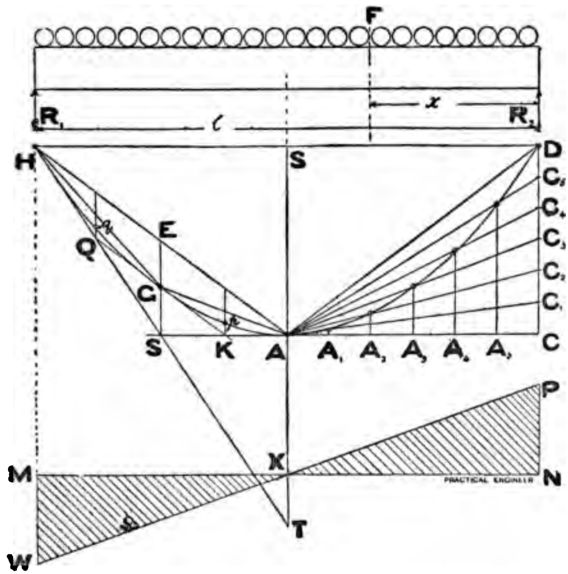

FIG. 10.

$+ 40 = y$, which represents another straight line. Give to x its least value, 4, then $y = 56$ tons-feet, the same result obtained just previously. Next give x its maximum value, namely, 10; then $y = 80$. Set off $A_2\, m = 10$, and $m\, H_2 = 80$, and join $H_2 E_2$. Carry this process on, and we find the bending moment under the load 5 tons is 72 tons-feet. Plotting $n\, K_2 = 72$, and joining $H_2 K_2$ and $K_2 B_2$, we obtain the bending moment diagram $B_2 A_2 E_2 H_2 K_2$, such that the

bending moment at any section of the beam, say Z, is represented by the ordinate $Z_1 Z_2$ of the bending moment diagram lying immediately below Z; also the shearing force there is represented by the ordinate $p\,q$ of the shearing force diagram.

In constructing the bending moment diagram, the student will find it well to always take the moments of the forces on the same side of the section, so as not to confuse the signs, the same as in the shearing force diagram. It will be noticed in fig. 8 that the boundary of the bending moment and shearing force diagrams only changes direction underneath the loads, and hence all that is required is to find the bending moment or shearing force under the loads, and plot the diagram from those quantities.

Before going on to the consideration of the construction of the bending moment diagram by a purely graphical method, we will take two more examples, one with a continuous load and the other with a mixed load.

In the first place, given a beam sustaining a uniformly distributed load of three-quarters of a ton per foot run, find the bending moment at any section along the beam, which is 20 ft. long.

The beam is shown in fig. 10, loaded uniformly, and supported at the ends. Then, taking the general case in which the load is w tons per foot run, and the total span of the beam is l feet, the bending moment at any section F distant x from the right-hand end is the sum of the moments of all the forces to the right of the section F, and therefore

$$= R_2 \times x - \text{moment of } x \text{ feet of load about F}$$

$$= \frac{w\,l\,x}{2} - w\,x \times \frac{x}{2}$$

$$= \frac{w}{2}(l\,x - x^2) = y, \text{ say} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If this equation be plotted after giving several values to x, the resulting curve will be found to be the parabola D A G H, whose vertex A lies underneath the centre of the span. The vertical intercept cut off by the parabola and the chord H D represents the bending moment at a section of the beam lying vertically over the intercept.

If the above equation (12) is not at once recognised as representing a parabola, it may be compared with the general equation of the second degree—

$$A\,x^2 + B\,x\,y + C\,y^2 + D\,x + E\,y + F = 0,$$

which represents an ellipse if $4AC$ is greater than B^2, a parabola if $4AC = B^2$, and a hyperbola if $4AC$ is less than B^2. In (12), $B = 0$, and $C = 0$; hence $4AC = B^2$, and hence the curve is a parabola. But perhaps it may be more easily seen by transforming the equation thus: Divide equation (12) by $-\frac{w}{2}$, add $\frac{l^2}{4}$ to both sides, and we have

$$\left(x - \frac{l}{2}\right)^2 = \frac{2}{w}\left(\frac{wl^2}{8} - y\right)$$

which is of the form $X^2 = cY$, where $X = x - \frac{l}{2}$ and $Y = \frac{wl^2}{8} - y$; which means that the vertex of the parabola lies at a point represented by $x = \frac{l}{2}$ and $y = \frac{wl^2}{8}$.

Now, knowing the curve is a parabola, and the position of the vertex, it is easier to describe the curve by graphic methods rather than to find the co-ordinates of different points on the curve and then plot them to scale. A very easy method is the following: Draw the base line HD parallel to the beam and equal to it in length. Bisect HD in S, and set off $SA = \frac{wl^2}{8}$. Through A draw AC parallel to SD, and through D draw DC parallel to SA. Divide AC and DC into the same number of equal parts (six in figure), and to each of the points of section C_1, C_2, &c., draw lines radiating from the vertex A. Now at each of the points of section A_1, A_2, &c., erect perpendiculars. The points of intersection of the perpendiculars with the corresponding radiating lines all lie on a parabola. Therefore, after obtaining the points, draw through them a curve; this is the required parabola.

It may be easily proved that the curve so formed is a parabola. A_4M is parallel to CC_4;* therefore from similar triangles we get

$$\frac{AA_4}{AC} = \frac{A_4M}{CC_4}$$

and as AA_4 is the same fraction of AC as CC_4 is of CD,

$$\frac{AA_4}{AC} = \frac{CC_4}{CD}$$

* The point M is the point of intersection of the parabola and the radiating line AC_4.

therefore
$$\frac{A_4 M}{C C_4} = \frac{C C_4}{C D}$$

Multiply both sides by $\frac{C C_4}{C D}$, and we have
$$\frac{A_4 M}{C D} = \left(\frac{C C_4}{C D}\right)^2 = \left(\frac{A A_4}{A C}\right)^2$$

the ratio to be found in a parabola.

Another short method of drawing a parabola is that given in the left half of fig. 10. It is a theorem in conics that the tangent cuts the axis at a point as far on one side of the vertex as the ordinate cuts it on the other side. Therefore S A = A T. Again, if two points such as H and A are taken on the curve, and the chord H A be drawn together with the tangents S H, S A, at H and A respectively, then the middle point G of the intercept S E, cut off from a line through the intersection S parallel to the axis, is a point on the curve. This process being carried further by drawing the chords H G and G A, and the tangent Q G K at G (which is parallel to H A), the middle points q and p of the intercepts parallel with the axis through the intersections Q and K are points on the curve. By having the tangents to the curve at a number of points it is possible to obtain a more exact shape of the curve when only a few points are taken. The area contained between the chord H D and the parabola is not cross-hatched, as in the previous case, as it would take away some of the clearness of the figure.

The shearing force at any section F, distant x from the right-hand end, being the sum of the forces to the right of F, will equal R_2 – the weight of that part of the load to the right of F $= \frac{wl}{2} - wx = y$, say. This is evidently a straight line figure, y being a maximum when $x = 0$ or $x = l$, and a minimum when $x = \frac{l}{2}$. Therefore put up the ordinate N P $= \frac{wl}{2}$ and M W $= -\frac{wl}{2}$. Join W P, and we get the shearing force diagram. In the above the numerical values for w and l can be substituted and the several quantities obtained.

In the third example we have a beam A C, fig. **11**, resting upon two supports situated at distances of 3 ft. and 7 ft. from the ends. A concentrated load of 7 tons is situated ... from the left support and 7 ft. from the right support.

A load of 5 tons is uniformly distributed over the two portions that overlap the supports.

The total amount of overlap = (3 + 7) ft., and 5 tons distributed uniformly over this length of beam would be at the rate of half a ton per foot run. To find the supporting forces, we have from the first law of equilibrium

$$R_1 + R_2 - 5 - 7 = 0,$$

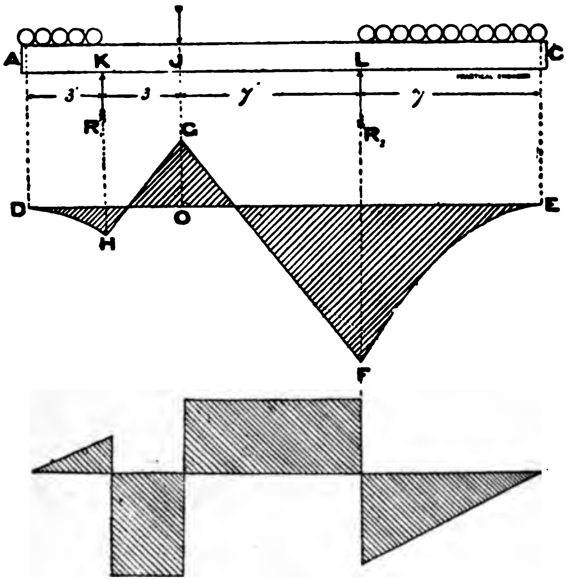

Fig. 11.

and from the second law (taking moments about L)

$$(3\tfrac{1}{2} \times 3\tfrac{1}{2}) - (7 \times 7) + (R_1 \times 10) - (1\tfrac{1}{2} \times 11\tfrac{1}{2}) = 0,$$

or $R_1 = 5{\cdot}4$ tons,

and hence $R_2 = 12 - 5{\cdot}4 = 6{\cdot}6$ tons.

The bending moment at any section between C and L distant x from C $= \dfrac{x^2}{4}$ tons-feet.

The curve $y = \dfrac{x^2}{4}$.

when plotted with the positive axis of y downwards is parabola E F. The bending moment at J = $(3\frac{1}{2} \times 10\frac{1}{2}) - (6{\cdot}6 \times 7) = -10{\cdot}95$ tons-feet. Put up in the negative direction O G = 10·95, and join G F. The remainder of the diagram is merely a repetition of the above.

At two points where the bending moment diagram boundary crosses the base line D E the bending moment is zero. These two points are situated at distances of 10 in. and 5 ft. 1½ in. respectively from H. The distinctive feature of a positive or negative bending moment is, that while one produces an upward sag, the other produces a downward sag in the beam.

The shearing force diagram, fig. 11, is easily obtained. Beginning at C, the sum of the forces increases with the distance until L is reached. Just beyond L the sum of forces to the right of section = $6{\cdot}6 - 3{\cdot}5 = 3{\cdot}1$ tons; hence the sudden jump upwards of the diagram underneath L. The shearing force remains constant until J is reached, and then the boundary moves downwards through a distance of 7. The remainder of the diagram is evident from preceding examples.

There is one fact which might be noticed at this stage of the investigation; it is that the ordinate of the bending moment diagram is a measure of the area of the shearing force diagram up as far as the ordinate. This may be easily verified by experiment, for take that portion lying under L C. The area of the shearing force diagram is the area of a triangle whose height is 3·5 and base 7, namely, 12¼. Again, the bending moment at J is − 9·45, and the area of the shearing force diagram lying under J C is $12\frac{1}{4} - (7 \times 3{\cdot}1) = -9{\cdot}45$.

CHAPTER IV.

Graphical Determination of Reactions and Bending Moment.*

Let us take the beam, fig. 12, loaded with three weights W_1, W_2, W_3 tons, as shown; the distance of the weights from the line of action of the supporting force R_2 being x_1, x_2, x_3 feet respectively, the total span being l feet. The arrows in the figures denote the lines of action of the several forces. Now, instead of naming a force by a letter near it, as is often done, we will here adopt the method first introduced by Professor Henrici and Mr. Bow, which is to

* See Appendix.

name a force or member by the two letters that occur on either side of it. For instance, between C and D we find the force W_2 ; hence in the upper portion of the figure the force W_2 may be designated by C D. Therefore, having put down the lines of action of all the forces acting *on* the beam

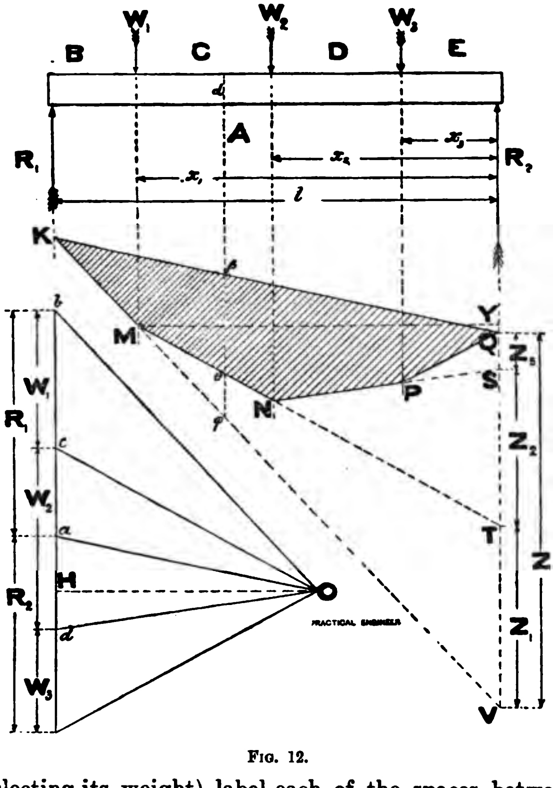

FIG. 12.

(neglecting its weight), label each of the spaces between a pair of forces ; thus the space between W_1 and W_2 has been labelled C, that between W_2 and W_3 labelled D, that between W_3 and R_2 labelled E, that between R_2 and R_1 labelled A, and that between R_1 and W_1 labelled B. Now, the

supporting force R_1 is designated A B because it lies between A and B ; or in passing from the space A to that of B we must cross the force R_1. The diagram so far drawn is purely a diagram of *lengths* or distances.

Now draw a polygon of forces for the beam. Starting at b, draw $b\,c$, $c\,d$, and $d\,e$ equal to and parallel to the forces B C, C D, and D E respectively ; that is, equal and parallel to W_1, W_2, and W_3. It will be here noticed that in the force polygon the lines representing forces are named by the letters at their extremities, which letters are the "italics" of the capital letters designating the position of those forces in the *length* diagram. This rule will be followed throughout these notes, and it will be found to possess many advantages. The line $b\,c\,d\,e$ is sometimes called the "line of loads," and, of course, need not necessarily be a straight line in all cases. Because all the forces acting on the beam are vertical, the polygon of *those* forces must necessarily be a straight line, or, more correctly, two straight lines coinciding with each other. Thus having begun at b and arrived at e, the only two remaining forces to be represented are R_2 and R_1, and as the beam is in equilibrium, the sum of the vertical components of the forces must be zero ; or, in other words, we must arrive back at the point b from which we started to draw the force polygon, and therefore $R_1 + R_2 = e\,b$. The first point to be decided, then, is how much of $e\,b$ represents R_1 and R_2 respectively.

Produce the lines of action of all the forces acting on the beam. Take *any* point O, and join O b. Now, through the space B of the *length* diagram draw a line parallel to O b, cutting the lines of action of the forces R_1 and W_1 in K and M. It may be drawn in any convenient position, but it is better to prevent any overlapping of the figures, if possible. Join O c, and through M draw a line parallel to it, passing through the space C, between the lines of action of W_1 and W_2, and cutting the latter in N. Join O d and O e ; then draw the parallel lines N P, P Q through the respective spaces D and E. All the lines through the spaces B, C, D, &c., are parallel to lines drawn from the pole O to the points b, c, d, &c. Now join Q K, and through the pole O draw a line parallel to Q K, cutting the line of loads in a ; then (as will be proved later on) $e\,a = R_2$ and $a\,b = R_1$. The shaded figure K N Q is called a *funicular* or link polygon. Produce the sides K M, M N, &c., of the funicular polygon, until they cut the line of action of R_2 in V, T, and S. Let V T, T S, and S Q be represented by Z_1, Z_2, and Z_3 respectively ; also let $Z_1 + Z_2 + Z_3 = Z$.

rom the second law of equilibrium we have

$$R_1 l - W_1 x_1 - W_2 x_2 - W_3 x_3 = O \quad . \quad . \quad (13)$$

Because the two triangles O b c, M V T are similar (the sides O b, b c, and c O being parallel to the sides M V, V T, and T M),

$$\frac{O b}{M V} = \frac{b c}{T V} = \frac{W_1}{Z_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Also, because the triangle b H O is similar to the triangle V Y M (M Y being horizontal), then—

$$\frac{O b}{M V} = \frac{O H}{M Y} = \frac{O H}{x_1} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Combining equations (14) and (15), we have—

$$\frac{W_1}{Z_1} = \frac{O H}{x_1}, \text{ or } W_1 x_1 = O H . Z_1 \quad . \quad . \quad (16)$$

Now, $W_1 x_1$ is the moment of W_1 about a point in the line of action of R_2 ; hence O H (a constant), multiplied by Z_1, is also the moment of W_1 about a point in the line of action of R_2. It will be well to notice here the relation existing between these quantities. Z_1 is measured in the line in which the point is taken, about which the moment is reckoned, and its magnitude is determined by being cut off by two lines of the funicular polygon produced, which are parallel to the two lines in the pole diagram, which are drawn from the pole to the extremities of the line representing the force which is producing the moment.

In the same way the moment of W_2 about a point in the line of action of R_2 is

$$W_2 x_2 = O H . Z_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and the moment of W_3 is

$$W_3 x_3 = O H . Z_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Substituting these values in (13), we have—

$$R_1 l - O H (Z_1 + Z_2 + Z_3) = O$$

or, $$R_1 l = O H . Z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Now, comparing (19) with (16), and reasoning conversely, we see that as Z is the intercept V Q, cut off by the lines K V and K Q, drawn from a point in the line of action R_1, parallel to O b and O a, therefore a b must represent R_1 in the same way that b c represents W_1. It is then evident that e a represents R_2. An easy rule by which to remember which supporting force is represented by either a e or a b is

that, whichever supporting force is near the space letter in the length diagram, the same force will be near the same letter in the force or pole diagram. For instance, R_2 is next to the space E; hence $a\ e$, which is next to e, will represent R_2.

From what has been so far said with reference to fig. 12, it is clear that the lines there drawn are the graphical representation of the laws of equilibrium as applied to the beam in question. But the laws of equilibrium are quite independent of the form of a body; hence the same construction must hold for *any* and *all* bodies in equilibrium—that is, the polygon of external forces or line of loads must *close* or return to the point from which it started; also the funicular (shaded) polygon must *close*.

It has now to be seen how the bending moment diagram is to be obtained. Returning to the interpretation of equation (16), let us take a point in the vertical line $\alpha\,\phi$, about which to take moments. Then the moment of W_1 about this point is evidently O H $\times\ \theta\,\phi$, because $\theta\,\phi$ is the intercept cut off on the line in which the point lies, about which the moment is taken by two lines emanating from a point in the line of action of W_1, parallel to the two lines in the pole diagram drawn from the pole to the extremities of the line representing W_1. In the same way the moment of R_1 will be O H $\times\ \beta\,\phi$. The bending moment at the section α is the sum of the moments of all the forces on the left of the section $=$ O H $\times\ \beta\,\phi\ -$ O H $\times\ \theta\,\phi =$ O H $\times\ \beta\,\theta$.

Now, $\beta\,\theta$ is the intercept on the vertical line through α, cut off by the boundary of the funicular polygon; and therefore the bending moment at any section is the vertical depth of the funicular polygon, multiplied by the horizontal distance O H. As this latter quantity is constant, the funicular polygon (shaded) is the bending moment diagram.

We have next to find the scale of this diagram. The beam was drawn to a scale of *lengths*, say p feet to the inch. The line of loads or force polygon was drawn to any convenient scale, say q tons to the inch. Similarly, the pole O was chosen at any convenient horizontal distance from the load line, say r inches. Now, O H is a line in the pole diagram, which is a *diagram of forces*, and therefore O H must be to the same scale as the line of loads—that is, q tons to the inch. Then the bending moment at

$$\alpha = \text{O H} \times \beta\,\theta = (r \times q) \text{ tons} \times (\beta\,\theta \times p) \text{ feet}$$
$$= (r \times q \times p)\,.\,\beta\,\theta\,.\text{ tons-feet} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Therefore the scale of $\beta\,\theta$ is $(r \times q \times p)$ tons-feet to the inch. Of course, $\beta\,\theta$ is here measured in inches.

All the lines in the upper portion of fig. 12 represent lengths, either real or imaginary; at the same time, all the lines in the pole diagram represent forces, either real or imaginary. For instance, the triangle $b\,c\,O$ in the pole diagram has three sides parallel to the three lines M K, M N, and the vertical through M. Also a force W_1, acting in one of these lines (the vertical through M), is represented in magnitude and direction by $b\,c$; therefore the other lines, c O, O b represent forces which, if applied at M in the directions M N and M K, would, with W_1, be in equilibrium. If, therefore, three strings were tied together so that their point of junction coincided with M, and two of them were attached to the points K and N such that their lengths were K M and M N, and a weight or force W_1 applied vertically to the third string, then the tensions in the two inclined strings would be represented by O b and c O respectively.

In the same way, the point N would be in equilibrium under the action of the three forces W_2, O c, and d O. Also at P, the forces W_3, O d, and e O would be in equilibrium. The point K would be in equilibrium under the action of the three forces $R_1 = a\,b$, b O, and O a. Similarly, at Q we should have $R_2 = e\,a$, a O, and O e in equilibrium. In this way, all the lines in the *pole diagram* would be used twice over, which is evidence of the equilibrium of the supposed structure; for it is equivalent to saying that equal and opposite forces have been found in each member. The strings may be replaced by a series of weightless rigid links, without disturbing the equilibrium; hence the name *link* or *funicular polygon* given to the shaded figure.

In this problem virtually lies the germ of graphic statics, and it is on this account it should be thoroughly understood before any further progress is attempted.

As an example of the method just described, let it be required to draw the bending moment and shearing force diagrams of the beam in fig. 13, which sustains a concentrated load of 5 tons at the right-hand extremity, and a load of 1·75 tons per foot run uniformly distributed over 4·9 ft. of the beam, beginning at a distance of 1·32 ft. from the left supporting force. The right supporting force R_2 is situated 4·21 ft. from the right-hand end.

Divide the uniformly-distributed load into, say, six equal parts, and consider it as equivalent to six concentrated loads acting through their respective centres of gravity. In the figure those loads are denoted by arrows. Now

label all the spaces between the forces as indicated in th
figure, and put down to scale the loads in the load line b

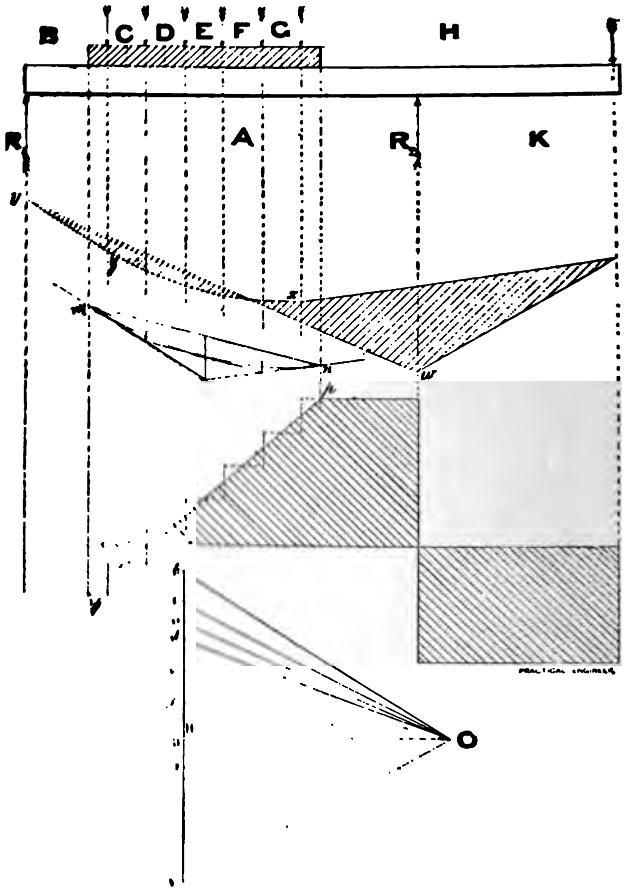

Fig. 18.

and join the extremities of the loads to the pole O. T

construct the funicular polygon, draw through the space B a line parallel to O b, cutting the lines of action of R_1 and B C in v and y. Then through the space C draw a line parallel to O c, and so on, until the points v and w remain to be closed by drawing the line $v\ w$. Then through O draw O a parallel to $v\ w$, and we have the line of loads divided at (a) into the two supporting forces, such that $R_1 = a\ b$ and $R_2 = k\ a$. The (shaded) funicular polygon is the bending moment diagram.

Because the left-hand load is uniformly distributed, we should naturally expect the contour of the diagram immediately under this load to be a portion of a parabola (equation 12); but having considered the load as broken up into a number of small loads, we were only approximating to the actual distribution, with the result that, instead of the parabolic arc $m\ n$, we have the series of chords between y and x, which very nearly coincide with it. The difference is so small as not to materially affect the result. Of course, the greater the number of pieces into which the distributed weight is divided, the more nearly does the diagram contour coincide with the parabola.

The shearing force diagram follows immediately from previous problems; the only point to be noticed is the substitution of the straight line $p\ q$ for the zigzag dotted line. The latter is what we should get if we assumed the distributed load broken up, as in the bending moment diagram; but it has been previously shown (fig. 11) that the contour of the shearing force diagram immediately under a uniform load is an inclined straight line, and the straight line $p\ q$ has been drawn on the strength of that previous knowledge, it being more easily accomplished and more accurate than the zigzag line. In the same way, had the parabola been as easily described as the chords between x and y, the parabola would have been drawn instead of the chords. It is merely a matter of facility.

CENTRE OF GRAVITY.

Another problem bearing directly upon the methods of reasoning just gone through is that of finding the centre of gravity of a number of bodies. We shall take only one case, namely, that of a set of bodies whose centres of gravity are in the same plane—that of the paper—from which others can be developed at will.

The centre of gravity of a system of bodies such as we have suggested is that point about which there is no

tendency of the system to turn, however it may be suspended; the relative distances of the bodies is supposed constant. It is then evident that the centre of gravity of the system is the point through which the resultant of a number of forces must act, those forces being equivalent to the weights of the bodies; and therefore it is that point through which a single force must act, which will maintain the weights in equilibrium. The result is exactly the same as if the thick horizontal line (fig. 14) represented the axis of a beam, and the several weights acted directly upon

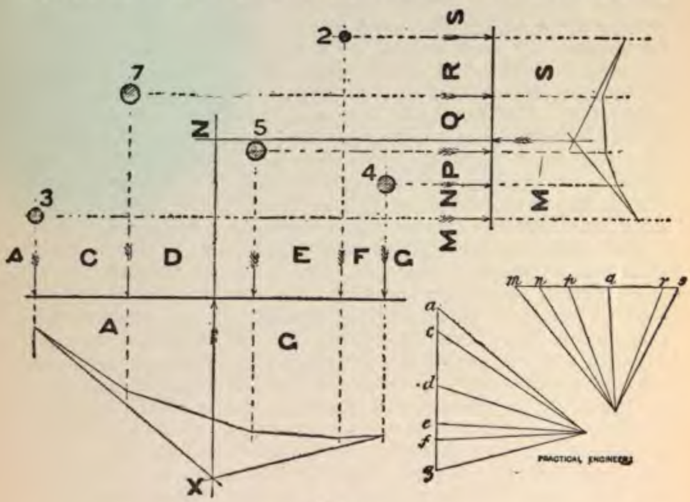

FIG. 14.

it along the vertical dotted lines through the centres of the bodies, the beam being supported by a single vertical force. The line of action of this force must pass through the centre of gravity of the weights.

Label the spaces between the assumed vertical forces and draw the load line $a\ g$, the numbers near the bodies representing their weights. Take any pole O, and draw the lines radiating from O to the extremities of the weights, and then draw the funicular polygon in the usual manner, each space being cut by a line parallel to the corresponding radiating line in the pole diagram. The two final lines

parallel to O a and O g intersect in a point X, through which the single vertical resultant force must pass. Hence this vertical line must contain the centre of gravity of the system.

As the centre of gravity is independent of the position of the system, we may turn the system round until lines that were previously vertical are now horizontal, and *vice versa.* We may then repeat the process and obtain another pole diagram, funicular polygon, and line—now vertical, but previously horizontal—in which the centre of gravity must lie. The intersection Z of this line with the previous line through X must be the centre of gravity of the system.

If the bodies do not all lie in the same plane, their centres can be projected on to a plane, and the projection of their centre of gravity on that plane can easily be found as above. The same can be done with respect to a second plane. These

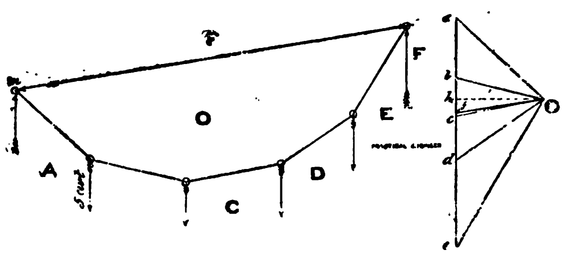

FIG. 15.

two projections of the centre of gravity are enough to determine its position in space. A plan and elevation will be found generally the most familiar projections to make.

FLEXIBLE LINKED STRUCTURE.

In fig. 15, m and n are fixed pins, to which are attached the ends of the link arrangement there shown. The problem is: Given the position of the several links, and the load at the end of the first link on the left being 5 cwt., What must be the loads at the other joints so as to maintain the set of links in the positions given? Also, What are the vertical supporting forces at each end, and the final equilibrating force between the two pins at m and n?

Put in the arrows denoting the several forces, and label the spaces as usual. Then put down as much of the line of loads as may be known (here the only known load is AB =

5 cwt.). The two forces which, together with A B, are in equilibrium, are the tensions in the two links A O and B O; and, as the stress in a link must act along the axis, the directions of the three forces are known, together with the magnitude of one of them. Hence a triangle can be drawn whose sides represent these three forces in direction and magnitude. Through the points *a* and *b* of the load line draw lines parallel to A O and B O respectively. These intersect at O. Through O draw lines parallel to O C, O D, O E, cutting the load line in *c*, *d*, and *e*. Then *b c*, *c d*, and *d e* represent the loads B C, C D, and D E, which must be suspended from the joints to make the chain take up the particular position that is given in the figure. It is evident that this set of links is in reality the funicular polygon previously spoken of, and that if *m*, *n* be joined, and through O a line be drawn parallel to O F, this line divides the line of loads into two parts, namely, the supporting forces *e f* and *a f*; and the line O *f* represents the stress exerted between the two pins to keep them apart in the positions shown. Also the lines O *a*, O *b*, &c., represent the stresses in the links O A, O B, &c., while the horizontal line O *h* (dotted) is the horizontal component of each of the stresses O *a*, O *b*, &c., in each of the links; therefore the horizontal component of the stress in any and all links of a linked structure, with vertical loads (such as in the figure), is constant.

SIMPLE HINGED TRUSS.*

In fig. 16 we have a truss supported at both ends, and loaded with one ton at each of three intermediate points. The several members are hinged together, without friction, all the forces being applied at the joints; hence the stresses in the different members must act along the axes of those members, as in fig. 7. It is evident from the symmetry of loading that half the total weight is supported by each end force. Label all the spaces between the forces, and between the members, and draw the load line *a d*. Bisect *a d* in *e*; then *d e* represents the right-hand supporting force, and *e a* that on the left. Now proceed to draw the stress diagram. Through *a* draw a line parallel to the member A F, and through *e* draw a line parallel to E F. The point *f* must lie in both of them, and hence it must be at their point of intersection. It will be noticed here that we have selected a couple of members, A F and E F, which have a common letter F, and therefore, as a space in the upper diagram corresponds to a point in the lower, it is evident that the

* *See also* Articulated Structures in the Appendix.

two lines parallel to A F and E F must intersect in *f*. This fixes the position of *f*. Now through *b* draw a line parallel to B G, and through *f* draw a line parallel to F G. These must intersect in *g*. Then through *c* and *g* draw lines parallel to C H and G H. These intersect in *h*. Proceeding in this way, the whole stress diagram is easily completed. Each line in the lower (stress) diagram represents the stress in the corresponding member of the frame; for example, the line *g h* represents the stress in the vertical

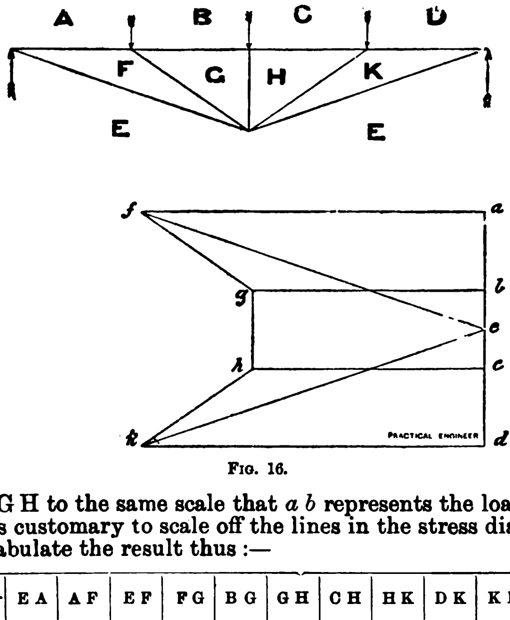

FIG. 16.

post G H to the same scale that *a b* represents the load A B.

It is customary to scale off the lines in the stress diagram, and tabulate the result thus :—

Member	E A	A F	E F	F G	B G	G H	C H	H K	D K	K E	E D
Stress in tons	1·5	4·4	4·6	1·7	2·9	1	2·9	1·7	4·4	4·6	1·5

In this particular truss it is easy to determine by inspection the nature of the stress in each member. First, take the vertical post G H. If it were removed, the force B C would make the two-hinged rods B G and C H sag downwards; therefore, as it prevents this, it must act

upwards on the force B C, or be in compression. Next take one of the intermediate members, such as F G. If F G were removed, the two rods A F and B G would sag downwards under the action of the force A B, and therefore F G resists this force, or is in compression. *If two hinged members lie in the same straight line, they have no power in themselves alone to resist a force at right angles applied at the joint.* This is evident from the fact that the resistance of friction at either joint is neglected, and as the stress in a hinged rod must act along the axis of the rod, neither of the stresses can have any component in a direction at right angles, and therefore neither of the rods can offer any resistance to a force at right angles. The stresses in F G G H, and H K being compressive, each of them tends to push the lowest joint further downwards, which tendency is resisted by E F and E K; therefore the stresses in each of these must be tensile. Again, at the remote hinge on the left side the horizontal component of the tensile stress in E F is resisted or balanced by the stress in A F; therefore this latter stress must be compressive. The stresses in each member of so simple a structure could be easily calculated, instead of being found graphically, but the methods of calculation will be left to some future occasion.

SIMPLE ROOF TRUSS. *

The roof truss in fig. 17 is assumed to be symmetrical, and symmetrically loaded with 2 tons at each of the intermediate joints; hence the supporting forces must be equal. As before, label all the spaces between the external forces, and put down the line of loads or force diagram $b\,g$. The point a of bisection of $b\,g$ corresponds to the space A in the frame diagram. Having completed the lettering of the spaces in the upper portion of the figure, the stress diagram may be proceeded with. Through a and b draw lines parallel to the members A H and B H, intersecting in h. Then through h draw a line parallel to H K, and through a draw a line parallel to A K. These intersect in h; therefore h and k are one and the same points, and hence the line $h\,k$ is of no length at all; that is, there is no stress in the member H K. As H K bears no stress, it can be removed without affecting the strength of the structure. In the same way, when the stress diagram is proceeded with, R S will be found to be a useless member. This result may have been anticipated from the remarks made when dealing with the previous truss. The members A H, A K, and H K are connected

* *See also* Articulated Structures in the Appendix.

together by a hinge, the two former being in the same straight line. This being so, there can be no stress in H K, because it is perpendicular to A H and A K.

It is now necessary to determine the kind of stress in each of the members of the structure. Starting with the point of application of the left supporting force A B, we

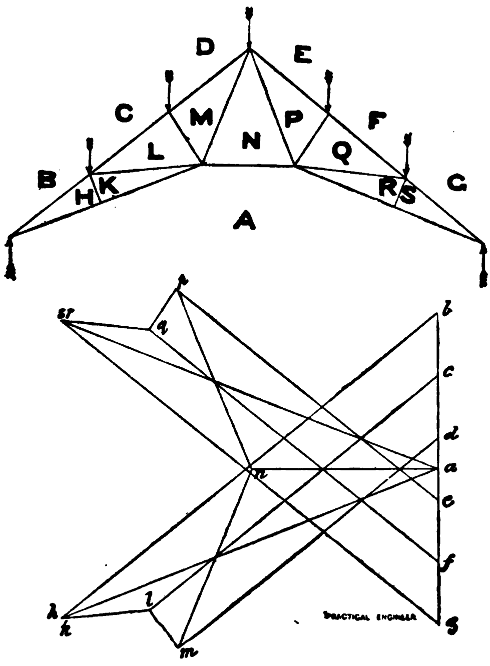

Fig. 17.

have at that point three forces in equilibrium, and therefore they should be represented in the stress diagram by the three sides of a triangle. This we find to be the case, the stresses being *a b*, *b h*, and *h a*. Also, if a number of forces are in equilibrium, and the directions of those forces be represented in the force diagram by arrow heads attached to the lines representing those forces, these

arrows will be found to follow each other completely round the force polygon in one direction. In the case in question we know that A B acts upwards, and following this direction in $a\,b$ round the triangle $a\,b\,h$, we see that the stress in B H and A H acts *at the point of application of* A B in the directions $b\,h$ and $h\,a$ respectively; that is, B H pushes the point of application, while A H pulls upon it. In this way we see that $b\,h$ is a compressive stress, and $a\,h$ a tensile stress. Now, because BH pushes at its lower end, it must also exert a push at the other end in the opposite direction, so as to maintain equilibrium in the rod itself; and similarly there must be an equal and opposite pull exerted by H A at its other end on the bars connected with it there.

At that point we have four other bars connected by a hinge, the direction of the action of one of them (H A) *at the point* being known. Following this direction round the polygon, a, k, l, m, n, a, we see that KL pushes the point of application, and the corresponding stress is, therefore, compressive. Similarly, the stress in L M is compressive, while that in M N and N A is tensile.

Tabulating the results of scaling off the stress diagram, we have :—

Member......	A B	B H	A H	K H	A K	K L	C L	L M	D M	M N	A N
Stress in tons.	5	−15·5	+13	0	+13	−2·8	−11·9	−1·5	−10·8	+6·3	+6

Only one-half of the stresses have been tabulated; the remaining half of the truss being similarly loaded, and symmetrically placed with respect to the former half, the stresses will be the same. This is obvious from a glance at the stress diagram.

WARREN GIRDER UNSYMMETRICALLY LOADED.*

The first case of unsymmetrical loading is that of the Warren girder, fig. 18, of which the lower boom contains five bays, and the upper boom four. It is loaded at three hinges in the lower boom with weights of two, five, and seven tons. The first part of the problem is to find the supporting forces. These may be calculated, but unless all the lengths happen to be integral numbers of feet, the process of calculation may be advantageously replaced by the graphical method previously given. Set down the

*See *Articulated* Structures in the Appendix.

line of loads $e\,b$, take any pole O, and draw the radiating lines of the pole diagram. Then draw the funicular polygon α, β, γ, δ, ϕ, with the closing line $\alpha\,\phi$. Through O draw O a parallel to $\alpha\,\phi$; then $e\,a$ is the magnitude of the supporting force E A on the left, and $a\,b$ that on the right.

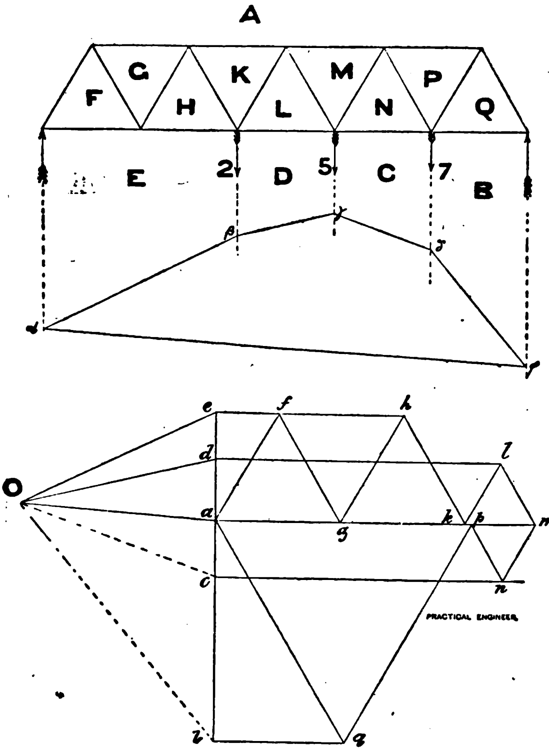

FIG. 18.

It may be well to notice here that, in beginning to draw the funicular polygon, we drew the line $\alpha\,\beta$ *anywhere* across the space E, parallel to O e. If the line of action of the force A E were unknown, the point α would be undetermined, and therefore it would be impossible to close the funicular

polygon as it now stands, and the supporting forces would remain undetermined. But, returning to the diagram (fig. 18), it is evident that the funicular polygon will still retain its form and dimensions if we move it bodily upwards, vertically, until the point a coincides with the point of application of the supporting force A E; and, by so doing, we ensure that a *shall be in the line of action of* A E, whatever be its direction. Therefore, when the direction of A E is unknown, *the first line of the funicular polygon must always be drawn through the point of application of* A E. Should the direction of the other supporting force be unknown also, it is evident the problem is insoluble, because the funicular polygon cannot be closed.

The stress diagram follows immediately without any difficulty, but the kind of stress in each member should be noted, as the unsymmetrical loading does not allow all symmetrically situated members to possess the same kind of stress. Following the general practice of denoting compressive stress by a negative sign, and tensile stress by a positive sign, because a compressive stress produces a negative strain or compression and a tensile stress produces a positive strain or elongation, we have the following table showing the magnitude and kind of stress in each member.

Member..	A E	E F	F G	G H	A G	E H	H K	A K	K L	D L	L M
Stress in tons ..	4·5	+2·7	+5·3	−5·3	−5·4	+8	+5·3	−10·7	−3	+12·2	+3
Member..	A M	M N	C N	N P	A P	P Q	A Q	Q B	A B	A F	—
Stress in tons ..	−13·7	+2 8	+12·2	−2·8	−11	+10·8	−10·8	+5·5	9·4	−5·3	—

UNSYMMETRICAL STRUCTURE.*

In fig. 19 we have a structure much resembling a heavy crane in appearance, loaded with a single weight of 5 tons, and supported at two points, one of which is α, the lowest point in the structure, the other point being the little square projection at the point of the arrow head in the line $\beta\ \phi$. As usual, the quantities to be determined first are the magnitudes and directions of the two supporting forces. From the nature of the structure in the figure, the supporting force C B is horizontal, and a is the point of application of the other support. The whole structure is maintained in

* See Articulated Structures in the Appendix.

equilibrium by three forces, and therefore these three forces must all pass through one point. The lines of action of the

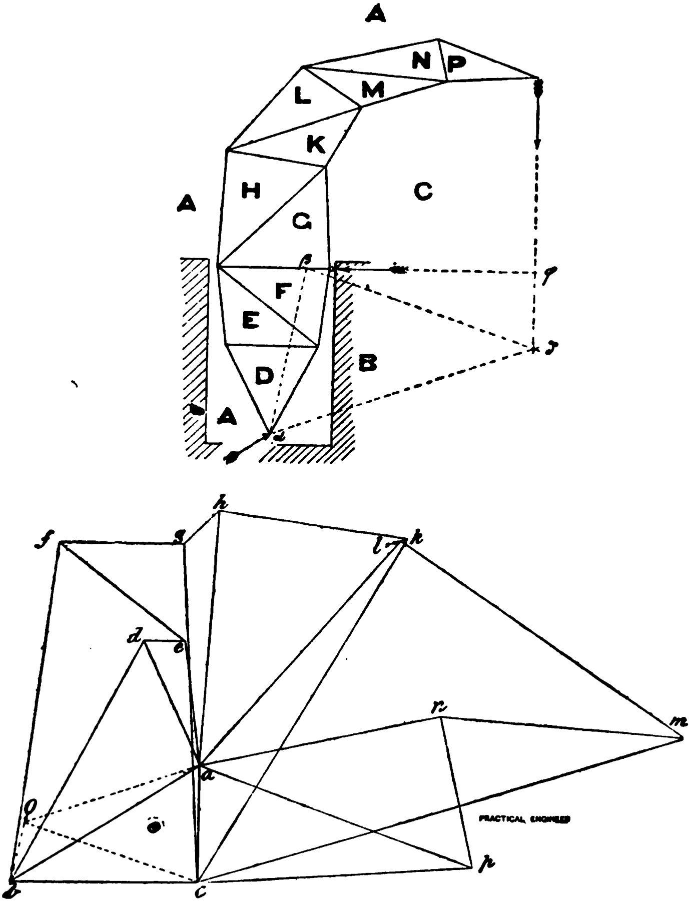

Fig. 19.

two forces, A C and C B, intersect at ϕ; therefore the third

force A B must also pass through ϕ. Hence join $\alpha\,\phi$, and this line represents the direction of the force A B. Having now the directions of the three forces and the magnitude of one of them, namely, A C, it is easy to construct the triangle of forces $a\,b\,c$.

This result may have been obtained direct from the use of the funicular polygon, thus: Set down the line of loads $a\,c$, and draw the line $c\,b$ of indefinite length. Take any pole O, and draw the radial lines O a and O c. As the magnitude and direction of the force A B are both undetermined, we must begin to draw the funicular polygon through the point α. The first line is drawn through the space A parallel to O a. The space A is bounded by the lines of action of the forces A C and A B, and therefore a line drawn through it must cut the lines of action of these two forces. Hence through α draw a line parallel to O a. It cuts the line of action of A C in δ. Through δ and across the space C draw the line $\delta\,\beta$ parallel to O c, cutting the line of action of the horizontal force in β. It will be seen that the link $\beta\,\delta$ of the funicular polygon is correct, thus: The next space to A is C, which is bounded by the lines of action of the forces A C and B C; hence the corresponding link in the funicular polygon must cut the two lines of action of A C and B C, which it does in δ and β.

Join $\beta\,\alpha$, and through O draw a line parallel to it, cutting $c\,b$ in b. The finding of the point b determines the magnitude of B C and the direction and magnitude of A B. Join $a\,b$; then $a\,b$ represents the magnitude and direction of the supporting force A B, while $c\,b$ represents C B. The stress diagram can now be drawn similar to previous figures. The results are here tabulated :—

Member.....	C B	B A	A D	D B	E D	E A	E F	F G
Stress in tons	7·8	9·3	+ 5·8	− 11·6	− 1·7	+ 5·3	+ 6·7	− 5·2

Member.....	F B	C G	H G	A H	A L	H K	K C	L K
Stress in tons	− 14·5	− 14·3	− 2·1	+ 10·8	+ 12·7	− 8	− 16·9	+ ·3

Member	C M	L M	A N	M N	N P	A P	C P
Stress in tons	− 21·4	− 14·7	+ 10·3	+ 10·4	− 6·5	+ 12·3	− 11·5

It will be noticed that each trapezium in fig. 19 contains a diagonal member. Each of those members may have joined the other two corners instead of those shown. The consideration which lead us to adopt one or the other of these, or both, at the same time must be deferred until we have investigated the elastic properties of materials. We shall then also be in a position to decide upon the most advantageous form of each individual member.

The junior student who may be unfamiliar with any of the mathematical passages in the succeeding portion of these notes, will do well to take the results upon trust and use them freely in any of the designs which may be given.

CHAPTER V.

Bending of Elastic Material.

Hitherto we have only dealt with the application of external forces to material in general, without investigating the behaviour of that material when subjected to those forces, or considering in any way the action between one particle and another inside the material itself. It is now proposed to deal with as much of this portion of the subject as will enable us to apply the graphic methods to structural design.

Experiment demonstrates that all material is more or less elastic, and that some of the metals possess this quality in a very marked degree. By elasticity is meant the tendency of a piece of material to return to its original condition after it has been deformed. Experiment also shows that there is a limit to the elasticity of a piece of material, and that this limit is reached long before rupture occurs; in other words, if a piece of material is deformed within certain limits—depending on the kind and nature of the material—it will return to its original condition when the deforming forces are removed; but if it is deformed to an extent which lies outside of the above limits, then the material will *not* finally return to its original condition; also, that if the applied forces are continually increased beyond the limit of elasticity, rupture will occur at some distance beyond that limit.

Now, *within the elastic limit* a definite relation exists between the deforming force and the amount of deformation

produced by the force. This does not hold good outside the elastic limit. This relation, which is deduced from experiment, may be stated thus: *The amount of deformation is proportional to the deforming force.* For instance, in a bar or rod which is subjected to a pair of equal and opposite forces, applied at its ends, tending to stretch it, the amount of elongation is directly proportional to one of the applied forces. If the forces are doubled, then the amount of elongation will also be doubled. The rod offers a resistance to each of the applied forces equal in magnitude to one of them. If, then, we take an imaginary section of the rod, the resistance offered by one part of the rod to the other part, across the imaginary section, is called the *total stress* over that section. The intensity of this resistance—that is, the amount of resistance per unit of area—is called simply the *stress* at that section.

The whole deformation of the rod when subjected to stress is called the *total strain*, but the intensity of the deformation per unit of length is called simply the *strain*. We may then write—

$$\text{Stress} = \frac{\text{total force applied}}{\text{sectional area}} \quad . \quad . \quad . \quad (21)$$

$$\text{Strain} = \frac{\text{total elongation}}{\text{original length}} \quad . \quad . \quad . \quad . \quad (22)$$

and

$$\left.\frac{\text{Stress}}{\text{Strain}}\right\} = \text{a } \textit{constant} \text{ for the same piece of material} \quad . \quad (23)$$

When the material is subjected to a pair of simple tensile or compressive forces, the above constant is approximately the same for the same *kind* of material; and thus we derive from experiment the following average values for it:—

Steel	29,000,000
Wrought iron	26,000,000
Cast iron	14,000,000
Copper	14,000,000
Phosphor bronze	13,000,000
Delta metal	12,000,000
Yellow brass	9,000,000
Wood	1,500,000

These figures are obtained when the stress is measured in pounds per square inch. The name which is given to the constant is "Modulus of elasticity," and often called "Young's modulus," to distinguish it from other moduli of

elasticity. Young's modulus is generally represented symbolically by the Roman character E, and will always be represented by it in these pages.

Of course, the above are only average numbers, as it must be manifest to the merest tyro that the hardness and ductility of the material will have something to do with its elasticity, and these quantities vary considerably; nevertheless, the numbers as they stand may be used with confidence in calculations which refer to the average material.*

When a piece of material—say a beam—is bent under the action of external forces, experiment shows that one surface is lengthened while the opposite surface is shortened by the action of the external forces. Now, if material is lengthened, it must be subject to tensile stress; and if shortened, compressive stress will be found in it. Again, if lines are drawn on the surface of the beam parallel with the axis, and in the plane of curvature, it will be found that while the beam is bent the lines which are lengthened or shortened the most are those nearest the top or bottom surface, and that the amount of elongation or contraction diminishes gradually as the lines are situated further away from the top or bottom surface; hence there will be some intermediate line which will be neither lengthened nor shortened. This line is the intersection of the surface containing all such lines in the beam with the plane of the paper. This surface is called the *neutral surface* of the beam. As the stress is proportional to the strain produced [equation (23)], the stress over the neutral surface must be zero, because there is no strain there.

Now, consider a small portion of a beam, fig. 20, which, when in the unbent state, is contained between the two parallel planes CD, NN_1; but when bent, this rectangular piece CNN_1D will assume the form CC_1D_1D, the line CN being lengthened to CC_1, and DN_1 being shortened to DD_1. A very thin longitudinal layer, HE_1, in the unstrained condition will, when strained, be lengthened to HH_1, the distance of the layer from the neutral surface being h. The elongation of the layer HE_1 is E_1H_1, and therefore, from equation (22),

$$\text{the strain} = \frac{E_1H_1}{HE_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Let f be the *stress* on the end of the layer—that is, the average force per square inch exerted upon it by that part

* See Appendix.

of the beam to the right of H_1, or to the left of H; then, from equation (23), we have—

$$E = \frac{\text{stress}}{\text{strain}} = \frac{f}{\frac{E_1 H_1}{H E_1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Now, $H E_1 = A A_1$, and $N A_1$ is parallel to A O; also the two triangles $E_1 A_1 H_1$ and $A O A_1$ are similar. Hence

$$\frac{E_1 H_1}{A A_1} = \frac{E_1 A_1}{A O} = \frac{h}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where R is the radius of curvature of the line $A A_1$, which

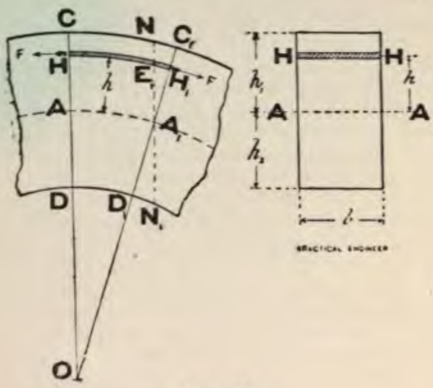

FIG. 20.

must equal A O, because two adjacent normals intersect at O, and, therefore, O is the centre of curvature of the curve $A A_1$. Substituting from (24) and (26) in (25), we get—

$$E = \frac{f}{\frac{h}{R}}, \text{ or } \frac{f}{h} = \frac{E}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Now, take an end view of the section of the beam whose width is . b. The end of the layer considered above is shown cross-hatched at H H. The area of this end is $b \,.\, d\,h$, where . $d\,h$. represents the small thickness of the layer; and the total stress over the end of the layer is $f \,.\, b \,.\, d\,h$.

Substituting the value of f found in equation (27), we get—

$$\text{Total stress over end of layer} = \frac{E}{R} . b . h . dh . \quad . \quad . \quad (28)$$

And as the cross-section of the beam is made up of the ends of these layers, the resultant stress over the whole cross-section will be the sum of the total stresses over the ends of the layers making up the cross-section ; *i.e.*, the sum of all the quantities $\frac{E}{R} \cdot b . h . dh .$ where h varies from $- h_2$ to $+ h_1$, the negative sign indicating measurement below the neutral axis A A. In the nomenclature of the calculus this is expressed as—

$$\text{Total stress over cross-section} = \int \frac{E}{R} . b . h . dh.$$

This total stress acts in a direction perpendicular to the plane of cross-section, and is therefore equal to the resultant horizontal force with which the portion of beam to the right of $C_1 D_1$ acts upon the portion to the left of it. As the beam is in equilibrium, and loaded with transverse forces, this resultant force must be zero, and consequently the right-hand side of the last equation must be zero.

Now, the first factor is a constant ; hence $\int b . h . dh$ must be zero. This is the analytical expression of the fact that the line from which h is measured passes through the centre of gravity of the cross-section.* This line A A is called the neutral axis of the cross-section.

Returning to equation (28), and the left-hand portion of fig. 20, we have: Moment of total end stress over layer about the neutral axis through $A_1 = F h = \frac{E}{R} b h^2 d h$, and the sum of all these moments over the whole cross-section must represent the moment of resistance of the beam at that section,

$$= \int \frac{E}{R} . b . h^2 . d h . = \text{M, say.}$$

The first factor is constant, and the remainder of the expression represents the geometrical moment of inertia† of the shaded strip about the neutral axis ; and therefore, after it is integrated over the whole section, the result will

* For proof of this theorem consult Minchin's "Statics," chap. x.; or Thompson and Tait's "Elements of Natural Philosophy," Art. 195.

† See article on "Moment of Inertia."

be the geometrical moment of inertia of the cross-section about the neutral axis; and if we write I for that quantity, the above equation becomes—

$$M = \frac{E}{R} I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28a)$$

As the moment of resistance is equal to the bending moment at the same section, we have, by combining (27) and (28*a*)—

$$\frac{f}{h} = \frac{M}{I} = \frac{E}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

DEFLECTION.

The radius of curvature of a plane curve, in terms of the co-ordinates x and y, is given by the expression—

$$R = \frac{\left[1 + \left(\frac{d\,y}{d\,x}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2\,y}{d\,x^2}} \, *$$

In general the amount of deflection of a beam is very small compared with its length, and therefore the inclination of the tangent to the curve of deflection at any point must also be small, and, necessarily, the tangent of the angle must be very small in comparison with unity, and therefore can be neglected when the ordinary approximations are required Neglecting $\frac{d\,y}{d\,x}$ in comparison with unity in the above expression, and substituting in equation (29), we have—

$$\frac{d^2\,y}{d\,x^2} = \frac{M}{E\,I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and

$$y = \iint \frac{M}{E \,.\, I}\, d\,x \,.\, d\,x \quad . \quad . \quad . \quad . \quad (31)$$

the bending moment M being expressed in terms of x. This is a convenient equation for some purposes. Let y_0 represent the maximum deflection; then, for a concentrated load W pounds, situated at the middle of the span whose whole length is l inches, I being constant—

$$y_0 = \frac{8}{384} \frac{W\,l^3}{E\,I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

* See Edwards' "Differential Calculus," chap. x.; or Williamson's "Differential Calculus," Art. "Curvature"; or Greenhill's "Calculus."

and for a total load of W pounds, uniformly distributed over the span, I being constant—

$$y_0 = \frac{5}{384} \frac{W l^3}{E I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Also for a concentrated load of W pounds, situated at a point of the span dividing it into two parts of length a and b, I being constant—

$$y_0 = \frac{W a^2 b^2}{3(a + b) E . I} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

With a single cantilever fixed horizontally at one end, and a single concentrated load at the other end—

$$y_0 = \frac{W l^3}{3 E I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

The same cantilever with a load W uniformly distributed—

$$y_0 = \frac{W l^3}{8 E I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

In cases where I is not constant, it will generally happen that it is some function of x which will be required to be put in before the integration is proceeded with.

Equation (31) may sometimes be conveniently written thus—

$$y = \iint \frac{f_0}{E h_0} \cdot \frac{M I_0}{M_0 I} \, d x . d x . \quad . \quad . \quad . \quad (37)$$

where the suffix 0 denotes the several quantities at the origin at the centre of beam. This equation is obtained by multiplying the right-hand side of (31) by

$$\frac{M_0 h_0}{I_0 f_0},$$

which is equal to unity [see equation (29)].

The result of performing the integration of this last equation for different conditions of loading and different kinds of beam is given on page 50 in tabular form. In every case $\frac{d}{2}$ (the half depth of beam) has been substituted for h. The deflection at the origin is denoted by y_0, and the deflection at any other point by y. The origin is taken where the tangent to the curve of deflection is horizontal.

A beam of uniform section, loaded with equal weights at equal distances x from the middle of the span, will give a maximum deflection at its centre—

$$y_0 = \frac{f\,l^2}{4\,\mathrm{E}\,d}\left[1 - \tfrac{1}{3}\left(1 - \frac{2\,x}{l}\right)^2\right]$$

Of course, in all the expressions above given for the deflection, it is assumed that the deflection is small, or that the

TABLE OF DEFLECTIONS. *

Form of cross-section throughout the whole length of beam.	Where supported.	How loaded.	Maximum deflection.
Uniform section........	At both ends.	At centre............	$y_0 = \frac{1}{6}\frac{f\,l^2}{\mathrm{E}\,d}$
Uniform section........	At both ends.	Uniformly distributed	$y_0 = \frac{5}{24}\frac{f\,l^2}{\mathrm{E}\,d}$
Uniform section........	At one end ..	At free end	$y = \frac{2}{3}\frac{f\,l^2}{\mathrm{E}\,d}$
Uniform section........	At one end ..	Uniformly distributed	$y = \frac{1}{2}\frac{f\,l^2}{\mathrm{E}\,d}$
Uniform depth, uniform strength	At one end ..	Uniformly distributed	$y = \frac{f\,l^2}{\mathrm{E}\,d}$
Uniform depth, uniform strength	At both ends.	Uniformly distributed	$y_0 = \frac{1}{4}\frac{f\,l^2}{\mathrm{E}\,d}$
Uniform width, uniform strength	At one end ..	At free end	$y = \frac{4}{3}\frac{f\,l^2}{\mathrm{E}\,d_0}$
Uniform width, uniform strength	At one end ..	Uniformly distributed	$y = 2\frac{f\,l^2}{\mathrm{E}\,d_0}$
Uniform width, uniform strength	At both ends.	At centre............	$y = \frac{1}{3}\frac{f\,l^2}{\mathrm{E}\,d_0}$
Uniform width, uniform strength	At both ends.	Uniformly distributed	$y_0 = \left(\frac{\pi}{4} - \frac{1}{2}\right)\frac{f\,l^2}{\mathrm{E}\,d^0}$
Uniform width, uniform strength, uniform depth	At both ends.	Anyhow	$y_0 = \frac{1}{2}\frac{f\,l^2}{\mathrm{E}\,d}$
Uniform section........	Bending moment constant throughout		$y_0 = \frac{1}{4}\frac{f\,l^2}{\mathrm{E}\,d}$

angle of the slope is never sensibly different from the circular measure of that angle.

For a full detailed account of the results in the above table, and how they are obtained, the reader is referred to Alexander and Thomson's "Applied Mechanics," vol. ii., pages 291 to 337.

* See Appendix.

The form of beam given in the last line of the above table is interesting. A constant bending moment may be produced by applying two equal and opposite couples to the ends; or it may be obtained by impressing upon the beam four equal forces, in pairs, near the extremities, similar to the forces upon the axle of a railway truck. As the bending moment is constant, together with the cross-section, and, necessarily, the moment of inertia, therefore the radius of curvature will also be constant—that is, the curve of deflection is a circle. This is apparent by substituting the constant quantities in equation (29). Also, because the half depth is constant, the stress in the extreme layers must also be constant throughout the length of beam.

BEAM OF UNIFORM STRENGTH, WIDTH, AND DEPTH.

As this kind of beam is very nearly related to the one just mentioned, and as it is much used in actual practice, we shall here give a simple deduction of the expression for its

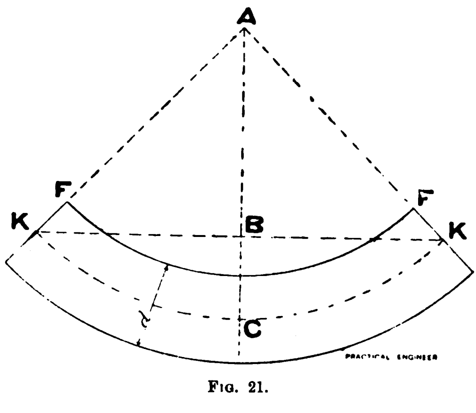

Fig. 21.

maximum deflection. It is evident that, as the depth and width are constant, the cross-section must be of the I or box form, and the thickness of the flanges must be varied to suit the bending moment. This is the ordinary form of built-up parallel wrought-iron girder. The half depth being constant, together with the intensity of stress over the flanges, equation (29) shows that the radius of curvature must also be constant—that is, the deflection curve is a circle, which is

identical with the previous case; and hence the maximum deflection

$$y_0 = \tfrac{1}{4}\frac{f\,l^2}{E\,d}.$$

This result may be obtained very simply, thus: In fig. 2 let l be, as usual, the length of the neutral surface $K\,K_1$, while l^1 is the length of the upper surface $F\,F_1$, and R is the radius of curvature of the neutral surface; also d is the depth of the beam. Because the deflection $B\,C = y_0$ is small, $K\,K_1$ is sensibly equal to l, and from Euclid III., 36, we have $2\,R \times B\,C$, sensibly equal to $B\,K^2$; or,

$$2\,R\,y_0 = \frac{l^2}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Also $$\frac{A\,K}{A\,F} = \frac{K\,K_1}{F\,F_1}, \text{ or } \frac{R}{R + \frac{d}{2}} = \frac{l}{l - l^1} \quad . \quad . \quad . \quad (39)$$

Again, $$\frac{l - l^1}{l} = \text{the strain} = \frac{f}{E}$$

which, if substituted in (39), gives

$$R = \frac{d\,.\,E}{2\,.\,f}.$$

Replacing R in (38) by its value just obtained, we have

$$y_0 = \frac{f\,l^2}{4\,d\,E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

The value of E, as used for built-up girders, is much less than for solid beams. Rankine gives 18,500,000 for built-up wrought-iron girders.

GRAPHIC METHOD OF OBTAINING DEFLECTION.

The deflection curve may be easily found graphically, without introducing the integration previously used. We shall first consider the case of a girder made up of a pair of flanges connected by a web, in which the flanges resist the bending action of the external forces upon the girder, the shearing actions being assumed to be entirely resisted by the web. The depth d in this case is assumed to be uniform throughout the length of the girder. Such a girder, in which the strained condition is very much exaggerated, is

shown in fig. 22. It is supported at both ends and bent by a uniform load, though the method holds equally well whatever the loading may be. The load being known, the bending moment at any section is easily determined, and when the area of the flanges is given, the stress intensity is at once obtained. For example, let M be the bending moment at any section, and f the stress in the flanges, while A is the area of each flange. There will be a total stress over each flange equal to A f tons, and these two forces, acting in opposite directions at a distance d. apart, form a

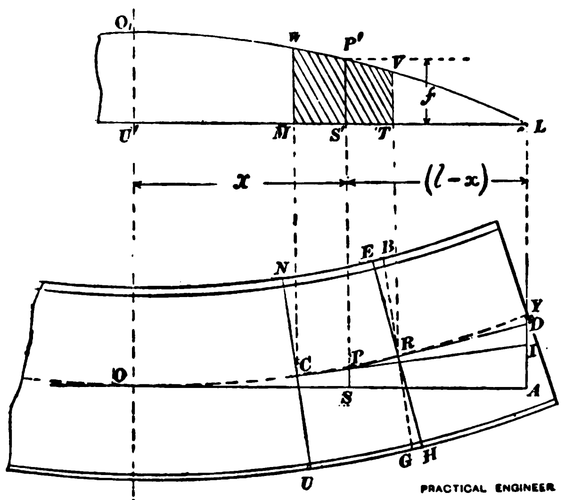

FIG. 22.

couple whose moment is A.f.d., which must be equal to the bending moment. Equating these quantities, we have

$$f = \frac{M}{A\,d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

from which a stress diagram may be plotted for the whole length of the beam. In this particular case it will be a parabola (upper portion of fig. 22) similar to the bending moment diagram, because d and A are both constant.

Now, consider a thin slice N.U.H.E. of the girder, fig. 22, which in the unstrained condition was the

rectangular piece N U G B. The bending has shortened N B by the amount E B, while it has lengthened U G by the amount G H. From equation (22), we have—

$$\text{Strain} = \frac{\text{total elongation}}{\text{original length}} = \frac{\text{G H}}{\text{U G}};$$

therefore

$$\text{U G} \times \text{strain} = \text{G H} = \text{R H} \times \text{circular measure of G R H} \quad (42)$$

The original unstrained position of H E was G B, and G B is parallel to N . U . ; therefore the angle between N U and E H is the same as the angle between E H and G B—that is, the angle G R H. This is then the angle between the two consecutive radii of the deflection curve, and as it is a theorem in geometry that *the angle between two radii equals the angle between the two tangents at the extremities of those radii*, therefore the angle G R H equals the angle between the tangents at R and C. These tangents cut the vertical through Y in D and I. Now, from equation (23), we have—

$$\text{Strain} = \frac{f}{\text{E}};$$

then, after substituting this in (42) above, we get—

$$\text{U G} \times \frac{f}{\text{E}} = \text{R H} \times \text{G R H} = \frac{d}{2} \times \text{G R H.} \quad . \quad . \quad . \quad (43)$$

or $$\text{U G} \times f = \frac{d\,\text{E}}{2} \times \text{G R H}; \text{ or } \frac{d\,\text{E}}{2} \tan \text{G R H},$$

because the deflection is so small that the tangent of the inclination is sensibly the same as the circular measure of the inclination. But U G = C R = M T; hence the left-hand side of the above equation is the area of the shaded part of the stress diagram, and the equation may be translated verbally thus : *The area of any portion of the stress diagram contained between two ordinates (such as* W M *and* V T) *is equal to the product of* $\frac{d\,\text{E}}{2}$ *and the tangent of the angle* (G R H) *between the tangents to the deflection curve at the two points immediately under the feet of the enclosing ordinates of the stress diagram.* And, similarly, the area O W L U_1 equals the product of $\frac{d\,\text{E}}{2}$ and the tangent of the angle which lies between the tangents to the deflection curve at O and Y.

The deflection of the end Y of the girder, due to the bending of the slice N . U . H . E., is I D, because, if the slice were not bent, then the tangent D R would coincide with I C. But D I = $(l - x)$ × the angle between the tangents at R and C = $(l - x)$ G R H, and after substituting the value of G R H from (43), we have—

$$\mathrm{D\,I} = (l - x) \times \frac{2}{d} \times \mathrm{U\,G} \times \frac{f}{\mathrm{E}} = (l - x)\,\frac{2 . \mathrm{C\,R.}}{d} \times \frac{f}{\mathrm{E}}$$

The *total* deflection of the end Y from the tangent O A at O is the sum of all the deflections such as D I, due to the bending of all the slices of the girder between O and Y ; and hence total deflection A Y = the sum of all the quantities,

$$(l - x)\,\frac{2\,\mathrm{C\,R}}{d} \times \frac{f}{\mathrm{E}}$$

where x may vary between nothing and O A. = l. This equation may be written thus—

$$\mathrm{A\,Y} = \Sigma \left[(l - x)\,\frac{2\,\mathrm{C\,R}}{d} \times \frac{f}{\mathrm{E}}\right]_{x=o}^{x=l.} \quad . \quad . \quad . \quad (44)$$

of which expression the product of the second and third factors represents the area of the shaded portion WM T V of the stress diagram multiplied by $\frac{2}{d\,\mathrm{E}}$ (which in this case is constant); while the first factor is the distance of the centre of the slice N U H E, and the shaded area W M T V from the vertical A L. We may now write equation (44) thus—

$$\mathrm{A\,Y} = \frac{2}{d\,\mathrm{E}} . \Sigma \left[(l - x) \times \text{area W M T V}\right]$$

$$= \frac{2}{d\,\mathrm{E}}\,\Sigma \left[\begin{array}{l}\text{moment of every shaded area about L, of} \\ \text{which the whole area L } \mathrm{U_1 O_1} \text{ is made up}\end{array}\right]$$

$$= \frac{2}{d\,\mathrm{E}} \times \text{moment of whole area L } \mathrm{U_1 O_1} \text{ about L} \quad (45)$$

$$= \frac{2}{d\,\mathrm{E}} \times \text{area L } \mathrm{U_1\,O_1} \times \text{distance of centre of gravity of L } \mathrm{U_1\,O_1} \text{ from L} \quad . \quad . \quad . \quad (46)$$

or, in other words, if a beam of length O A were fixed at the right-hand end as a cantilever, so that its neutral surface at that end is horizontal, and loaded with a load whose intensity is represented by the ordinates of the stress diagram, the new diagram formed by multiplying the

bending moment at any point (due to this imaginary load) by $\frac{2}{d\,E}$, and setting up this quantity as an ordinate from the base line O A, is that whose bounding lines are O A, A Y, and the dotted curve O P Y the trace of the neutral surface of the bent beam in the vertical plane. O Y is the curve of deflection. The deflection, measured from the tangent O A at a distance from the tangent point O equal to x, is S P, which, from the above, must be the moment of the area $O_1\,U_1\,S_1\,P_1$ about S_1, multiplied by $\frac{2}{d\,E}$.

It will be noticed that the deflection at any point is measured from, and perpendicular to, the *tangent* to the beam at the point O, and the point O may be taken arbitrarily, though it is generally advantageous to locate it immediately over a point of support or in the middle of the beam. We may now formulate this method of finding graphically the deflection curve of a bent beam.

Draw a diagram, showing the intensity of the stress in the flanges over that portion for which the deflection curve is required; then draw the diagram of bending moment for this imaginary load, treating that part of the beam under consideration as a cantilever fixed at the end remote from the tangent point. The ordinate to this curve (from the tangent) multiplied by $\frac{2}{d\,E}$ *gives the deflection.* The slice N U H E is, of course, assumed to be indefinitely thin in the direction C R; the reason why it appears so thick in the figure is that if it were taken smaller the several lines could not be distinctly seen.

The above method is very fully discussed by Professor Claxton Fidler in his treatise on "Bridge Construction," and its chief advantage lies in the fact that the construction is the same whatever may be the method of loading the beam.

In the above example we have selected a beam in which the direct tensile and compressive stresses have been resisted by the flanges only, while the web resists the shearing action. Since in a solid girder the stress at points intermediate between the neutral surface and the outside fibres is a function of the maximum stress in the outside fibres, and since the resultant of all those elementary stresses between those limits is a function of the maximum stress, it is evident that the above investigation also holds for solid beams, but the expression which then represents

the deflection will be the one above, multiplied by some constant, which will depend upon the form of cross-section.

If in equation (44) $d\,x$ be substituted for C R, and f is replaced by its value in terms of x, and then the integration be performed, the result obtained will be identical with the second case given in the table of maximum deflections. In general, the whole curve of deflection is hardly ever needed; it is merely the deflection at some cardinal point where it is either a maximum or a minimum, and these points often occur such that when the direction of the tangents are known, the required properties follow immediately.

We will now show that the *tangent of the inclination of a tangent line to the bending moment curve at any point is a measure of the shearing force at that point.* Let E F, fig. 23, be a beam supported by and fixed horizontally in the wall at F, and let it be uniformly loaded with w lbs. per foot-run. The bending moment at H will be $\frac{w\,x^2}{2}$ pounds-feet. Setting down R S in the usual way perpendicular to O C and equal to the bending moment at H, and repeating the operation at different points along E F, we obtain the bending moment diagram O S D C R, of which O S D is a parabola, whose vertex is at O. Draw a tangent to the parabola at S, cutting O E in T, and draw the abscissa S N. Then, from a property of the parabola N O = O T.* The inclination of the tangent T S to the horizontal O R or N S is the angle N S T, and

$$\tan N S T = \frac{T N}{N S} = \frac{2 O N}{N S} = \frac{2 R S}{x} = \frac{2 \times \frac{w\,x^2}{2}}{x}$$

$$= w\,x = \text{load on E H.}$$

$$= \text{sum of all forces acting on beam to the left of H}$$

$$= \text{shearing force at H.} \quad . \quad . \quad . \quad . \quad (47)$$

The same reasoning holds good whatever the loading may be.

With the aid of the above theorem we may go a step further, and show that *two tangents to the deflection curve intersect at a point immediately below the centre of gravity of*

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* See Wilson's "Solid Geometry and Conic Sections," p. 99, or any treatise on conic sections.

that part of the stress intensity diagram which is situated immediately above that part of the deflection curve contained between the points of contact of the two tangents and the curve.

In fig. 23 let the stress intensity be constant all along the beam ; then it can be represented by an ordinate of the dark area above the beam. The deflection at any point H being the moment of that portion of the dark area between E and H about H, multiplied by $\frac{2}{d\,E}$. It equals RS.

The same reasoning applies to all kinds of loading. Let the centre of gravity of the stress diagram be at G, distant $\bar{x}$ from F. Then it has been previously shown that if we

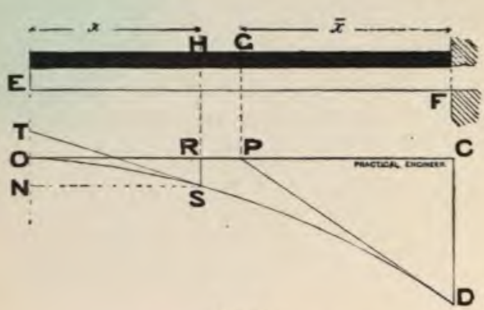


FIG. 23.

imagine the stress diagram to be divided into a very large number of small parts, whose areas are represented by W_1, W_2, W_3, &c., and whose centres of gravity are situated at distances x_1, x_2, x_3, &c., from F, that $(W_1 + W_2 + W_3, + \&c.)\ \bar{x} = W_1 x_1 + W_2 x_2 + W_3 x_3 + \&c.,$

$$= \text{moment of whole dark area about F.}$$

$$= \text{C D}$$

and $$\bar{x} = \frac{\text{C D}}{\text{whole dark area}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Again, the tangent to the curve at D is P D, and the tangent of its inclination to O C is $\frac{\text{D C}}{\text{C P}}$. But by equation (47) this is

equivalent to the shearing force at F, which equals the whole area of the stress diagram ; therefore we have

$$\frac{\mathrm{D\,C}}{\mathrm{C\,P}} = \text{whole area of diagram.}$$

Comparing this equation with (48), we see that $\bar{x} = \mathrm{C\,P}$, or *the tangent at* D *intersects the tangent at* O *immediately under the centre of gravity of that part of the stress diagram between* D *and* O.

Before proceeding to an example, there is one other relation to be shown to exist between the stress diagram and the deflection curve, and it is that the portion of the deflection curve immediately under a section where there is no bending moment is straight, and generally is the position of a point of inflexion. This may be shown in more ways than one. First, if there is no bending moment at any particular section, the beam will remain unbent there ; and if it were originally straight, it will continue to remain straight. If there is no bending moment, there will be no direct tensile or compressive stress on the flanges. Second, as a point of no bending moment is a point of no flange stress, this point will happen where the resultant stress diagram outline crosses the base line ; and, taking the signs into consideration, it will be the point where the stress changes through zero from positive to negative, or *vice versa* (from tension to compression, or compression to tension). Beginning at any ordinate of the stress diagram, the area of the diagram will gradually increase as we move towards the point of zero stress, and will be a maximum when that point is reached. After the point is passed the negative area will be subtracted from the positive, and the sum of the areas must necessarily decrease. But the sum of the areas is represented by the inclination of the tangent to the stress curve [equation (47)]; hence the tangent will be more and more inclined as the point of zero stress is approached. At that point the inclination will be a maximum, and as the point is receded from the inclination gradually decreases. This point of zero stress is then a point where the curvature changes from concave to convex, or *vice versa*, and that is called a point of inflexion. At the extremities of a beam which is not *fixed* in any way, but only supported, the beam is straight, but the end points are not points of inflexion, as the curvature does not change from positive to negative.

To emphasise these several relations, we will take any

arbitrary stress diagram, with its corresponding deflection curve, and point them out. In fig. 24, let F K G Z H J E be the stress intensity diagram described upon the base line E Z, such that the ordinates to the curve measured above E Z represent positive stresses, and below negative stresses. The corresponding deflection curve is T P S Y M. Now consider any portion T P S of the deflection curve. The corresponding part of the stress diagram is that which is contained between the two ordinates E F and H G. The tangents to the deflection curve at the extremities of the stress diagram under consideration are T N and S N. The tangent of the angle R N Q between them equals the area E F K G H J of the stress diagram multiplied by $\frac{2}{d\,E}$; also,

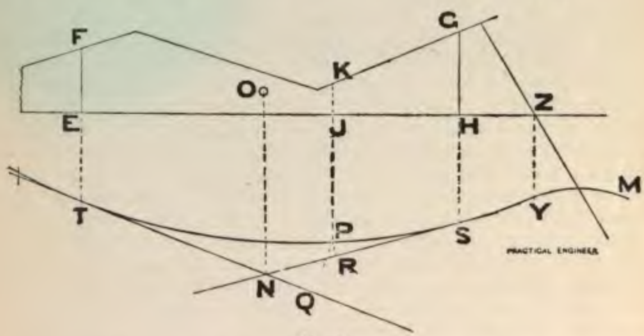


Fig. 24.

because the centre of gravity of the area E F G H is at O, the tangents at T and S will intersect at N immediately below O. The stress at Z being zero, the deflection curve immediately under it at Y will be straight, and the curvature there changes from concave upwards to concave downwards in passing through Y; therefore there is a point of inflexion there.

Again, any ordinate P Q to the deflection curve at P, measured from the tangent at T, equals the moment of the area E F K J about J K multiplied by $\frac{2}{d\,E}$; and the ordinate P R, measured from the tangent S R at S, equals the moment of the area H G K J about J K multiplied by $\frac{2}{d\,E}$.

In every case the area in question lies immediately above that portion of the deflection curve between the tangent point and the point whose ordinate is required, and its moment is taken about the point at which the ordinate is required.

CHAPTER VI.

Areas and Centres of Gravity of Parabolic Segments.

As it is sometimes useful to know the area of a segment of a parabola, or the position of its centre of gravity, these are here given. In fig. 25, O P is a portion of a parabola whose axis is O R, and vertex at O. The axis of x is here

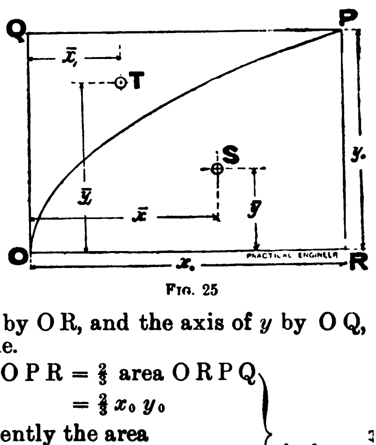


Fig. 25

represented by O R, and the axis of y by O Q, R O Q being a right angle.

$$\left.\begin{aligned} \text{The area } OPR &= \tfrac{2}{3} \text{ area } ORPQ \\ &= \tfrac{2}{3}\, x_0\, y_0 \\ \text{and consequently the area} & \\ OPQ &= \tfrac{1}{3} \text{ area } ORPQ \\ &= \tfrac{1}{3}\, x_0\, y_0 \end{aligned}\right\} \quad \ldots \quad (49)$$

If $\bar{x}$ and $\bar{y}$ denote the co-ordinates of the centre of gravity S of the area O P R, then

$$\left.\begin{aligned} \bar{x} &= \tfrac{3}{5}\, x_0 \\ \bar{y} &= \tfrac{3}{8}\, y_0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (50)$$

Also if $\overline{x_1}$ and $\overline{y_1}$ denote the co-ordinates of the centre of gravity T of the area O P Q, then

$$\left.\begin{aligned}\overline{x_1} &= \tfrac{3}{10}\, x_0 \\ \overline{y_1} &= \tfrac{3}{4}\, y_0\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (51)$$

In fig. 26, the area

$$\left.\begin{aligned}&\text{H K P R} = \tfrac{2}{3}\,[x_2\, y_2 - x_1\, y_1] \\ &\text{while the area} \\ &\text{K P N} = \text{area H N P R} - \text{area H K P R} \\ &\qquad = \tfrac{1}{3}\, x_2\, y_2 - x_1\,(y_2 - \tfrac{2}{3}\, y_1)\end{aligned}\right\} \quad . \; . \; . \qquad (52)$$

The co-ordinates of the centre of gravity G of the area H K P R are

$$\left.\begin{aligned}\bar{x} &= \tfrac{3}{5}\,\frac{x_2^{\frac{5}{2}} - x_1^{\frac{5}{2}}}{x_2^{\frac{3}{2}} - x_1^{\frac{3}{2}}} \\ \bar{y} &= \tfrac{3}{8}\,\frac{y_2^{\,4} - y_1^{\,4}}{y_2^{\,3} - y_1^{\,3}}\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (53)$$

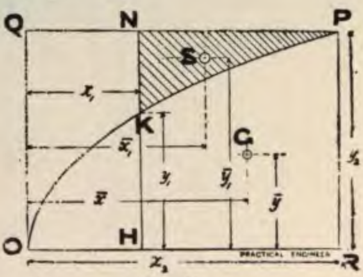


FIG 26.

The co-ordinates of the centre of gravity S of the area K P N are

$$\left.\begin{aligned}\overline{x_1} &= \tfrac{3}{10}\,\frac{x_2 + 2\,x_2\,x_1^{\frac{1}{2}} + 3\,x_1\,x_2^{\frac{1}{2}} + 4\,x_1^{\frac{3}{2}}}{x_2^{\frac{1}{2}} + 2\,x_1^{\frac{1}{2}}} \\ \text{or,} \qquad &= \tfrac{3}{10}\cdot\frac{x_2}{y_2^{\,2}}\cdot\frac{y_2^{\,3} + 2\,y_1\,y_2^{\,2} + 3\,y_1^{\,2}\,y_2 + 4\,y_1^{\,3}}{y_2 + 2\,y_1} \\ \text{and } \overline{y_1} &= \tfrac{3}{4}\cdot\frac{(y_2 + y_1)^2}{(y_2 + 2\,y_1)}\end{aligned}\right\} \quad (54)$$

Should the segment of the curve P K be nearly straight, the centre of gravity S may be found without much error by considering the figure P N K as a triangle.

BEAM FIXED AT ONE END AND SUPPORTED AT THE OTHER.

We shall now take as an example the case of a beam of double-tee section, uniformly loaded, of uniform depth and section, fixed horizontally at one end, and supported at the other. Such a beam is shown in fig. 27. If the support at E were removed, we should have a simple cantilever, uniformly loaded, and the bending moment at any section distant x from the free end would be represented by the expression $\frac{w x^2}{2}$, where w is the load per unit of length.

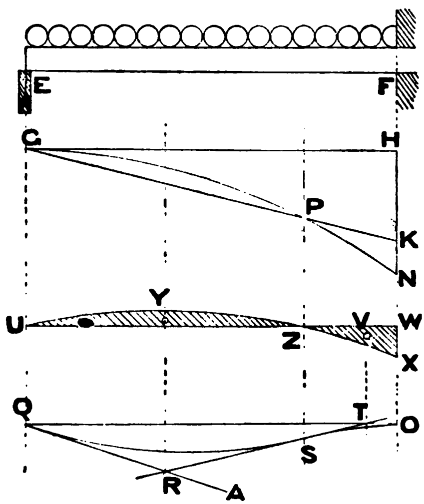


FIG. 27.

From this it is clear that the bending moment can be represented graphically by the ordinate to a parabola, whose axis is vertical, vertex at G, and concave downwards.

The moment of resistance of the beam at any section is $A f_1 d$, where A is the sectional area of one flange, d the depth between the centres of flanges, and f_1 the average stress over the flanges. Equating this expression to the bending moment, we have—

$$f_1 = \frac{w x^2}{2 A d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54a)$$

or the bending moment diagram also represents the intensity of stress in the flanges. Set down H N to represent the stress intensity in the flanges at F. If this is not known—and it is not necessary that it should be known just at present—we may set down H N of any length; but, at the same time, it will represent to some scale the stress f_1 at F due to the simple uniformly-loaded cantilever.

By what has been previously said, and equation (45), the deflection of the free end E below the horizontal E F is given by the moment of the area H N P G about G, multiplied by $\frac{2}{d\,E}$.

Let δ be this deflection, then—

$$\delta = \text{area HNPG} \times \text{distance of its centre of gravity from G} \times \frac{2}{d\,E}$$

$$= \tfrac{1}{3}\,.\,GH \times HN\,. \times \tfrac{3}{4}\,GH\,. \times \frac{2}{d\,E}$$

$$= \frac{GH^2 \times HN}{2\,d\,E} \quad \ldots \ldots \ldots \quad (55)$$

To produce the same result as the beam given in fig. 27, we must now apply a single upward force at E to bring the free deflected end of the cantilever back to the same level as F, or, in other words, the single force must be such that, if applied to the unloaded beam, would produce an upward deflection equal to δ. The bending moment diagram of a beam loaded at one end and fixed at the other is triangular with the vertex at the loaded end, and hence the stress diagram will also be triangular, similar to H K G. The upward deflection δ of the free end will be equal to the area H K G × distance of its centre of gravity from G × $\frac{2}{d\,E}$.

or

$$\delta = \tfrac{1}{2}\,.\,GH \times HK \times \tfrac{2}{3}\,GH. \times \frac{2}{d\,E}.$$

$$= \tfrac{2}{3}\,\frac{GH^2 \times HK}{d\,E}. \quad \ldots \ldots \ldots \quad (56)$$

And if we call the downward deflection positive and the upward negative, the above must be prefixed with the negative sign.

On referring to equations (32) to (36), it will be noticed that the deflections are directly proportional to the load producing them, and hence double the load will produce double the deflection, while treble the deflection must be produced by treble the load, other things being equal. In the same way, if δ_1 be the deflection due to a certain load W_1, and δ_2 that due to another load W_2, then $\delta_1 + \delta_2$ will be the deflection produced by the load $W_1 + W_2$. A glance at the table of deflection will show that the stress is proportional to the deflection, and hence the same rule holds good when the stresses are substituted for the loads.

Returning to fig. 27, the actual deflection of the point E will equal the sum of the deflections due to the two methods of loading; but this actual deflection is zero, E being on the same level as F; hence we have from (55) and (56)—

$$\frac{G\,H^2 \times H\,N}{2\,d\,E} - \frac{2}{3}\frac{G\,H^2 \times H\,K}{d\,E} = 0,$$

and therefore $H\,K = \frac{3}{4}\,H\,N$.

That is, having drawn the parabolic diagram H N P G, the problem is to draw a triangle whose moment about G shall equal the moment of the parabolic diagram about G. To secure this relation, H K must from the above be three-quarters of H N. As one diagram represents positive stress, while the other represents negative stress, the actual stress is represented by the difference of the two diagrams, which has been plotted separately in fig. 27, and is the shaded portion, with U W as base line.

The centre of gravity of U Y Z is at Y, and that of Z W X at V. Immediately under V will be found the intersection of the tangents to the deflection curve at O and S, while under Y will occur the intersection of the tangents at Q and S. The point S being below the point of no stress Z, it will be a point of inflection in the deflection curve, and hence the tangent at S will intersect the tangent at O in T, and the tangent at Q in R. The whole area of the stress diagram, namely, area U Y Z − Z W X (the latter being negative) multiplied by $\frac{2}{d\,E}$ is the tangent of the angle between the tangents to the deflection curve at its extremities = tan T Q R, Q R being the tangent at Q, and Q O the tangent at O. The area U Y Z multiplied by $\frac{2}{d\,E}$ = tan S R A, because R S is the tangent at S, and Q R the tangent at Q, the points Q and S being immediately

under the extremities of that part of the diagram U Y Z. Also tan S T Q = area Z W X multiplied by $\frac{2}{d\,E}$. Having obtained T, the point of intersection of the tangent at S, and that at O, and knowing the inclination of S T from above, R T can be at once drawn, intersecting the vertical through Y in R. Join Q R, and the third tangent is found. If the vertical scale of the deflection curve is not too large, the curve is fairly represented by describing arcs of circles which touch the tangents at Q, S, and O. Such a curve is shown in fig. 27. At S there is no bending moment, and the part S O to the right supports Q S and its load; therefore the portion Q S may be treated as a simple beam supported at the ends, when the point S is known. In this case S is situated at three-quarters of E F from E. This may easily be found arithmetically, thus: Find what value of x will make the value of y the same for the straight line G K as for the parabola G N. The equations to the parabola and straight line are respectively—

$$x^2 = 4\,a\,y, \text{ and } x = m\,y.$$

To determine the values of a and m, put in some known values of x and y in each equation. Those are known at the right-hand end, where $y = \text{H N} = f_1$ for parabola, and $y = \text{H K} = f_2$ for the straight line; also $\text{G H} = l$. Putting these in the above equations, we get $\frac{l^2}{4f_1} = a$ and $\frac{l}{f_2} = m$, and after substituting these and solving the simultaneous equation, we find that $x = \frac{f_2}{f_1}\,l$. But $f_2 = \frac{3}{4}f_1$; therefore $x = \frac{3}{4}\,l$.

If Q Z is three-quarters of the whole length, it sustains three quarters of the whole load, and as the part Q S is similar to a simple beam supported at each end, half its load is upon each end, and therefore there are three-eighths of the whole load to be supported at Q, and consequently five-eighths at O.

The actual stress at F

$$= \text{W X} = \tfrac{1}{4}f_1 = \frac{w\,l^2}{8\,\text{A}\,d}$$

from equation (54a).

The actual stress at Y

$$= \left[\text{R} \times \tfrac{3}{8}l - \tfrac{3}{8}\,w\,l \times \frac{3\,l}{16.}\right] \times \frac{1}{\text{A}\,d}$$

$$= \tfrac{9}{128}\,\frac{w\,l^2}{\text{A}\,d.}$$

When dealing with the case of continuous girders, we shall require to return to the graphic theory of deflection; but for the present we must proceed with other matters connected with the deformation of elastic material.

CHAPTER VII.

BENDING ACCOMPANIED BY DIRECT LONGITUDINAL FORCE.

WHEN dealing with the phenomenon of bending in the previous pages it was assumed that the external forces which were applied to accomplish the bending were in all cases perpendicular to the same surface of the material, which was itself initially plane. The result of the action of these external forces on the material was found to be a stress in it of two kinds, viz., tensile and compressive, in a direction parallel to the longitudinal axis of the material, and a shearing stress in a direction perpendicular to the axis. The former is sometimes called an induced stress, and equation (27) shows that its intensity is proportional to its distance from the neutral or longitudinal axis; therefore it can be represented in intensity by a diagram whose boundaries are straight lines; *e.g.*, in fig. 28 E F represents the trace of the plane of cross-section, and S the trace of the neutral axis, the depth of the beam being E F. Then, if E P represent the compressive stress in the material situated in the upper surface of the beam, and F Q the tensile stress in the lower surface, P S Q is a straight line, and the stress at any point J in the cross-section distant x from the neutral axis is given by the length J Z. For

$$\frac{\mathrm{J\,Z}}{\mathrm{J\,S}} = \frac{\mathrm{E\,P}}{\mathrm{E\,S}} = \frac{f}{h}.$$

Hence, if J S represents the distance from the neutral axis, J Z represents the stress due to bending.

If, in addition to the bending forces, we stretch the beam longitudinally by a pair of equal and opposite forces applied at the ends, we produce another tensile stress throughout the whole of the beam of intensity equal to the whole force divided by the area of cross-section. This is generally termed a direct stress, and is independent of the induced stresses due to bending. In fig. 28, set off E G equal to this

direct stress, and draw G H parallel to E F, cutting P Q in T. The total maximum tensile stress in the material is now P E + E G = P G, and the corresponding maximum compressive stress is F Q − F H = Q H ; while the position of no stress is at T—that is, by the superposition of a direct stress, the neutral axis has been displaced to a point nearer the surface, the amount of displacement being S F − T H. The stress at any intermediate point being given by the length of an ordinate drawn from and perpendicular to G H, and cut off by P Q.

A rod in tension, the stretching force being applied at a point which is not the centre of gravity of the cross-section.—The rod is shown in fig. 29. The force F is applied at P, which is situated at a distance a from the centre of gravity G. At G apply two opposite forces, each equal to F in magnitude (upper portion of figure). These two forces

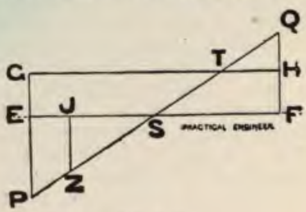


FIG. 28.

being themselves in equilibrium, will not in any way affect the rod. The upper force on the left, together with the right-hand force, form a couple whose moment is F a. Hence we have acting on the rod a couple tending to bend it, and a single longitudinal force F tending to stretch it. Using the previous notation, with f_t representing the direct tensile stress, and f_b the induced bending stress, we have—

$$f_t = \frac{F}{A} \text{ and } f_b = \frac{M}{I} h \text{ from equation (29).}$$

Setting these out, as in fig. 28, we get maximum tensile stress

$$= f_t + f_b = \frac{F}{A} + \frac{M}{I} h,$$

and maximum compressive stress

$$= f_b - f_t = \frac{M}{I} h - \frac{F}{A}.$$

If in these expressions we substitute for the moment of inertia I, its value $A\rho^2$, ρ being the so-called radius of gyration; also F.a. for M, whilst calling the maximum stress f the above expressions

may be written in one, thus— $f = \frac{F}{A}\left(\frac{a\,h}{\rho^2} \pm 1\right)$

the upper sign being used for maximum tensile stress, and the lower one for maximum compressive stress. The first

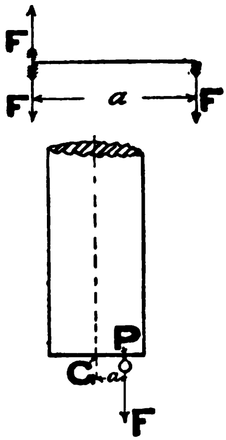


Fig. 29.

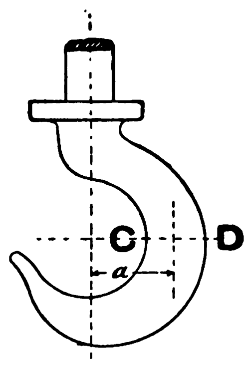


Fig. 30.

factor of the right-hand side is f_t. Therefore we may write—

$$f = f_t\left(\frac{a\,h}{\rho^2} \pm 1\right) \quad . \; . \; . \; . \; . \; . \; . \quad (57)$$

From this it is seen that the stress in a rod increases with an increase in the displacement of the point of application of the external force from the centre of gravity of the cross-section.

For a practical example of this we will take the draw bar or crane hook, fig. 30, and assuming the area of cross-section to be the same in each of the three cases—when the section is a rectangle, a circle, and a trapezium—to find the maximum safe load that can be applied to the hook if the limiting stress be 5 tons per square inch.

* See Appendix.

The section whose neutral axis is situated furthest from the line of action of the impressed forces is that at C D. Let the area of this section be 2 square inches, and the safe load in the three cases be W_1, W_2, and W_3 respectively. In the first and third case let the length of the cross-section C D be 3 in., and in the latter let the thickness of the hook at C be 1 in.; then the thickness at D must be about $\frac{3}{8}$ in. As the area of cross-section is the same in each case,

$$f_t = \frac{W}{A} = \frac{W}{2}$$

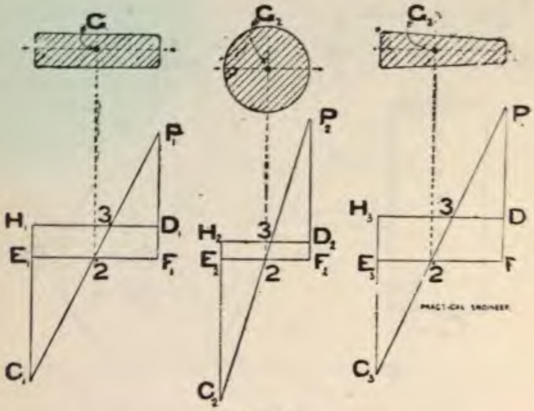


FIG. 31

Case I. (rectangle):

$$f_b = \frac{M}{I} h = \frac{W_1 \, a \, h}{\frac{1}{12} b \, d^3}$$

$$= \frac{W_1 \times 2 \times 1\frac{1}{2} \times 12}{\frac{2}{3} \times 27} = 2 \, W_1$$

(in every case a = the distance from the centre line of the impressed forces to the middle point of the line C D, and is taken in each case as 2in.);

and $\quad 5 \text{ tons} = f = f_b + f_t = 2\,W_1 + \frac{W_1}{2} = 2\frac{1}{2}\,W;$

hence $\quad W_1 = 2 \text{ tons}.$

Case II. (circle):

$$A = \frac{\pi}{4} d^2 = 2;$$

therefore
$$d = \sqrt{\frac{8}{\pi}}$$

$$I = \frac{\pi}{4} r^4 = \frac{\pi}{4} \frac{d^4}{16} = \frac{1}{\pi}$$

$$f_b = \frac{M h}{I} = \frac{W_2 \times a \times \frac{d}{2}}{\frac{1}{\pi}} = 5\,W_2$$

$$f = f_b + f_t = 5\,W_2 + \frac{W_2}{2} = 5 \text{ tons};$$

hence
$$W_2 = {\cdot}92 \text{ ton.}$$

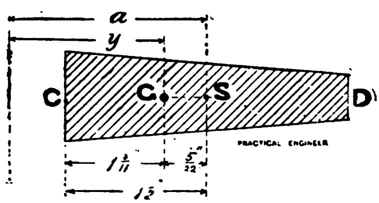


Fig. 32.

Case III. (trapezium):

The distance x of the (centre of gravity), neutral axis of the cross-section from C, is given by the equation—

$$x = \frac{d}{3} + \frac{d\,b_1}{3(b_1+b_2)} = 1\tfrac{3}{11} \text{ in.}$$

where b_1 and b_2 are the lesser and greater thicknesses respectively, and d the depth, 3 in. The distance x may have been found more quickly graphically if the cross-section section were drawn to scale.

$$I_3 = d \times \frac{b_2{}^2 + 4 b_1 b_2 + b_1{}^2}{36 b_2 + b_1} = 1{\cdot}96.$$

$$f_b \text{ (compressive)} = \frac{W_3 \times 1\tfrac{17}{22} \times 1\tfrac{8}{11}}{1{\cdot}96} = 1{\cdot}56\,W_3$$

The quantity $1\frac{17}{32}$ is the distance in inches of the centre of gravity of cross-section from the line of action of the impressed forces—

$$= y = a - \mathrm{G\,S}\ (\text{fig. } 32),$$

$$= a - (\mathrm{CS} - \mathrm{CG}) = 2 - (1\tfrac{1}{2} - 1\tfrac{3}{32}) = 1\tfrac{17}{32}.$$

Also,

$$f_b\,(\text{tensile}) = \frac{\mathrm{W}_3 \times 1\frac{17}{32} \times 1\frac{3}{32}}{1{\cdot}96} = 1{\cdot}15\,\mathrm{W}_3$$

$$f\,(\text{tensile}) = f_t + f_b\,(\text{tensile}) = \frac{\mathrm{W}_3}{2} + 1{\cdot}15\,\mathrm{W}_3 = 5 \text{ tons};$$

therefore $\mathrm{W}_3 = 3{\cdot}03$ tons.

The stresses in each of the above cases are plotted to scale in fig. 31; similar letters denoting similar stresses in each of the three cases.

$$f\,(\text{compressive}) = f_b\,(\text{compressive}) - f_t = 1{\cdot}56\,\mathrm{W}_3 - \frac{\mathrm{W}_3}{2}$$

$$= 1{\cdot}06 \times 3{\cdot}03 = 3{\cdot}2 \text{ tons per square inch.}$$

In this instance the material in compression is not being strained to the extent which is permissible with regard to safety, and hence it is possible, by arranging the shape of the cross-section, to increase the stress f (compressive) to the limit, and thus produce a more economical form of hook. This can be accomplished by decreasing the thickness of cross-section at the thin end, and increasing it at the other end, while the area is maintained constant. Then the final stress f (compressive) will be represented by $\mathrm{P}_3\,\mathrm{D}_3$, and f (tensile) by $\mathrm{H}_3\,\mathrm{C}_3$, while $\mathrm{H}_3\,\mathrm{E}_3$ represents the load F divided by the area of cross-section.

Strut.—We shall next consider the case of a rod or member subjected to a pair of equal and opposite forces, acting in a direction parallel to the axis of the rod tending to shorten it; that is, the forces will be compressive.

As this part of our subject is extremely interesting, and as the behaviour of a strut depends upon so many different phenomena, some of which are anything but determinable with exactitude, the author considers that he is to a certain extent justified in discussing this question at some length.

In structures, the majority of struts are either fixed at both ends to something more or less rigid, or fastened at the ds with pins. These two cases correspond to some extent h the two main cases of horizontal beams, namely, with

the ends fixed in walls or abutments, or merely supported at the ends. In the latter case the whole of the resistance to flexure is supplied by the material of the beam, while in the former the rigid abutments render a considerable amount of resistance in addition to the elastic resistance of the simple beam.

The term *ends fixed* is in general applied to struts in which the direction of the axis at the ends is in the same straight line with the axis of the strut in the unstrained condition, and that the directions of the ends of the strut do not vary during the existence of any straining action on the strut. In fig. 33 we have a horizontal strut with flanged ends, the flanges being so large that no change of direction of the axis at the ends can take place during flexure. The same result would be obtained if the strut were prolonged at both ends and riveted to the vertical cross-beam.

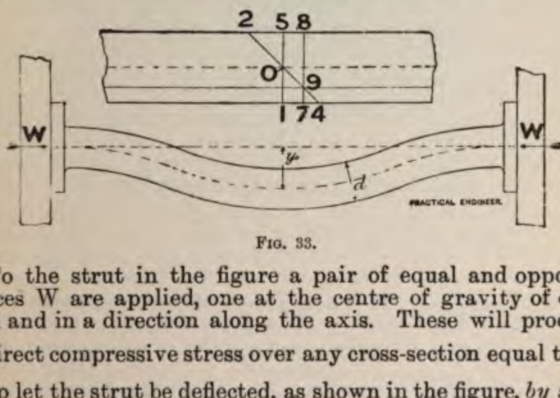


Fig. 33.

To the strut in the figure a pair of equal and opposite forces W are applied, one at the centre of gravity of each end and in a direction along the axis. These will produce a direct compressive stress over any cross-section equal to $\frac{W}{A}$

Also let the strut be deflected, as shown in the figure, *by some cause or other*. We shall then have a bending moment equal to W y_0 acting on the strut tending to increase the deflection y_0, which bending moment will be resisted by the moment of resistance of the beam, if the whole is in equilibrium. We then get from equation (29) the stress due to *bending only*,

$$= f_b = \frac{M}{I} h = \frac{W\, y_0\, h}{I} = \frac{W\, y_0 \frac{d}{2}}{A\, \rho^2}$$

where the area of cross-section is A, and the radius of gyration ρ. On referring to the table of deflections

previously given, it will be noticed that, however the beam is supported or fixed, the maximum deflection can be expressed as some multiple of $\frac{f\,l^2}{d}$, and for any particular stress as some multiple of l^2 divided by d. Therefore we may write: Maximum deflection y_o = some constant $\times \frac{l^2}{d}$ Substituting in the above equation, and writing f_c for W ÷ A, we have

$$f_b = c\,f_c\,\frac{l^2}{\rho^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (58)$$

Where c is a constant depending on the kind of material, and, as we shall discover later on, the way in which the strut is fixed or supported at the ends, and upon the form of cross-section.

Then

$$f = f_c + f_b = f_c + cf_c\,\frac{l^2}{\rho^2} = f_c\left(1 + c\frac{l^2}{\rho^2}\right)$$

which we may write as

$$f_c = \frac{f}{1 + c\,\frac{l^2}{\rho^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (59)$$

or, after multiplying both sides by A, we have

$$W = \frac{A\,f}{1 + c\,\frac{l^2}{\rho^2}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (60)$$

In this expression f is the maximum safe compressive stress permitted in the material, when designing a strut.

It has been previously mentioned that the constant ratio of stress to strain only holds good up to the elastic limit, beyond which no such law exists, and that fracture takes place when a piece of material is strained considerably beyond the elastic limit, so that such formulæ as those above, which are constructed on the assumption of perfect elasticity, cannot be expected to give the exact load that will produce failure in any particular strut.

When material is strained beyond the elastic limit, a permanent change takes place in the dimensions of the material, so that when the straining forces are removed, the material does not return to exactly the same form as that previous to being strained. This permanent deformation is generally termed a "set," and is due to a flow of the material

under excessive stress, in a manner something similar to a very viscous fluid. The actual maximum stress in a strut just previous to failure is always greater than that corresponding to the elastic limit of the material, and hence it must remain to a certain extent an indeterminate quantity in the above equations, although it is possible to locate it approximately between certain limits.

In fig. 33 a stress diagram has been sketched immediately above the strut, in which f_c is represented by the line 5.8., and f_b by 2 5. The maximum crushing stress f is then given by 2.8, while the maximum tensile stress is equal to 7.4. on the same scale. In the figure, 7.4. appears to be about one-fourth of 2.8., so that should the resistance of the material to tension be less than one-fourth the resistance to compression, this strut would fail by tension instead of by compression. We have such a material in cast iron, and it is a fact that between certain limits cast-iron struts often do fracture by tension. In any case the constant c is found by inserting known values of the other symbols from experiments in equation (60) and solving for c*. Generally f and c are both unknown, and hence two equations have to be solved simultaneously to obtain them. These two equations will contain quantities derived from at least two distinct experiments. Now, it is well known that two pieces of material of the same kind, manufactured in the same way, and treated throughout in the same manner, often differ very materially in physical properties; *e.g.*, the modulus of elasticity E of the two pieces may be widely different, and it is not uncommon to find two pieces of the same material, cut from different parts of the same block, to give a difference of modulus of 10 per cent when strained in the testing machine. In obtaining equation (59) we made use of an expression for the deflection which involved the modulus of elasticity E, and therefore, unless the modulus were the same in the two experiments from which f and c were derived, and still the same in all the material which it is proposed to utilise in the construction of struts, it cannot be expected that the above formula will give results which are more than approximately true.

Again, if we grant that the moduli of elasticity are the same throughout, there is no guarantee that the whole of the material will be of the same hardness. The result is that the f of one piece of material will not be the same as f of another piece; and hence the load sustained by one

* See page 78.

strut may not be the load which another strut can support (deduced from the same formula), although everything is apparently the same.

In the outset we assumed that the axis of the strut was perfectly straight, and that the load was applied at the centre of gravity of the end. It is seldom these two conditions are ever fulfilled, and when they are it is generally by mere chance. It will be shown in what immediately follows that a very minute initial "camber" (*i.e.*, deviation of the axis from the straight line), or a displacement of the point of application of the impressed forces from the centre of gravity of the ends, will cause a very considerable diminution in the load a strut will sustain.

In any single experiment all these elements of deviation from ideal circumstances may occur, and in a very large number of similar experiments there must be—according to the laws of probability—a few instances in which they all conspire to wreck the strut; while again there must be, approximately, a similar number in which they conspire to strengthen the column. There will be at the same time a much larger number in which the elements of deviation conspire in a less degree to wreck and to strengthen the column respectively, and, finally, there must be a very large majority of cases in which a sort of levelling-up action occurs; *i.e.*, where a discrepancy tending to strengthen is neutralised by another discrepancy tending to wreck the strut. It will be these last experiments that give results most near to those predicted by the expression derived from assuming real conditions.

Unfortunately such a vast number of experiments have never been carried out, partly on account of the expense involved, and partly on account of the time and labour that would be required. However, a few disjointed sets of experiments have been conducted for special purposes by different observers, and, so far as they go, they support the views advanced above. In fig. 34 the results of some of these have been plotted on a diagram, in which vertical ordinates represent the direct crushing stress f_c, while abscissæ represent the ratio $\frac{l}{\rho}$ (length to radius of gyration). The whole of the results in this figure represent struts whose ends were either fixed or flat, or so well bedded that they may be taken as fixed. Some experiments represented by the black dots were carried out by Hodgkinson many years ago upon solid square pillars of wrought iron

and are those from which Gordon derived the unknown quantities f and a in equation (60), which has since been known as Gordon's formula. An account of these experiments may be found in Mr. Stoney's "Theory of Stresses in Girders and Similar Structures," page 417.

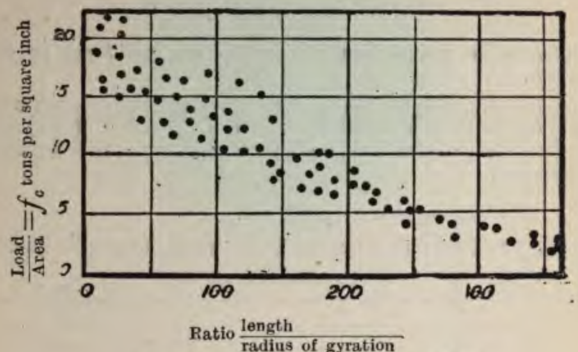


FIG. 34.—WROUGHT IRON. ENDS FIXED.

Some dots denote experiments on hollow wrought-iron cylinders, also carried out by Mr. Hodgkinson, and others represent the results of Mr. Christie's experiments on angle and tee iron carried out in America, and published in the Proceedings of the American Society of Civil Engineers, 1884. A *résumé* may also be found in the Journal of the Institution of Civil Engineers, vol. lxxvii., p. 396. The result of a number of experiments will also be found in the same publication, vol. xxx., which were conducted by Fairbairn and Clark.

The quantities in Gordon's formula are often taken from the average of a number of experiments. Upon a little consideration, it must be apparent that such a proceeding is not altogether justified by a reference to the diagram. Thus, a strut which is designed with average data may be much too weak if constructed of the same material as those represented on the lower side of the group. The ultimate object in design is to secure sufficient strength under all circumstances, and hence it would appear more reasonable to adopt data obtained from struts which indicate an average minimum of strength.

This idea has been put forward by Professor Claxton Fidler in a paper on the "Practical Strength of Columns," in the Proceedings of the Institution of Civil Engineers, vol lxxxvi.

If preferable, we may express the radius of gyration ρ in terms of the least transverse dimension d, thus—

$$d = n\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

where n is a constant depending on the form of cross-section.

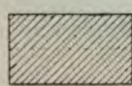
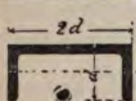
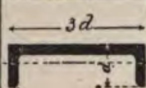
In fig. 35 will be found the corresponding value of the constant n in the equation $d = n\,\rho$ for some of the cross-sections in general use, where d is the *least transverse dimension.*

TABLE OF CONSTANTS TO BE USED WITH GORDON'S FORMULA. EQUATIONS 59 OR 60.

Material	f per sq. in. at yield point.	Young's Modulus. E per sq. in.	C.			
			$r=4$ both ends fixed.	$r=2.$ one end fixed.	$r=1.$ both ends round.	$r=\frac{1}{4}.$ see fig. 40.
Wrought Iron.	14 tons 31,500 lbs.	12,000 tons 27,000,000 lbs.	$\frac{1}{34,500}$	$\frac{1}{17,200}$	$\frac{1}{8,600}$	$\frac{1}{2,150}$
Structural Steel.	18 tons 40,000 lbs.	14,000 tons 31,000,000 lbs.	$\frac{1}{31,200}$	$\frac{1}{15,600}$	$\frac{1}{7,800}$	$\frac{1}{1,950}$
Cast Iron.	30 tons 67,200 lbs.	6,200 tons 14,000,000 lbs.	$\frac{1}{8,000}$	$\frac{1}{4,000}$	$\frac{1}{2,000}$	$\frac{1}{500}$
Pine.	·8 ton 1,800 lbs.	500 tons 1,220,000 lbs.	$\frac{1}{27,000}$	$\frac{1}{13,500}$	$\frac{1}{6\,250}$	$\frac{1}{1,560}$

In connection with the above table, it must be explained that cast iron has no definite yield point in the same way that the other materials have, but the constants there given agree very well with the result of experiment, and in consequence can be used with confidence in the design of a strut.

TABLE OF VALUES OF n FOR USE IN THE EQUATION
$$d = n\,\rho.$$

SECTIONS	N	SECTIONS	N	SECTIONS	N
	3·2		3·5		4·9
	3·5		4		4·9
	3·8		4·3		4

PRACTICAL ENGINEER

FIG 35.

CHAPTER VIII.

INVESTIGATION OF THE STRENGTH OF AN IDEAL STRUT.

WE will now attack the problem of the strength of a strut by a method slightly different from that just previously used.

Taking the general case, assume that the impressed forces W are applied at points situated at a distance (a) from the centre of gravity of the ends respectively, as shown in fig. 38. The axis of the strut will then be represented by the bow-shaped curve $O\,O_1\,O_2$, the ends of the strut being guided only in the line $O\,O_2$. In this case the ends are not *fixed*, as in that just considered, so that now the axis at the extremities will not coincide with the originally unstrained axis. Assume that the strut is made of *homogeneous elastic* material, and in the unstrained condition was perfectly straight. Let the deflection at any point P distant x from the origin be y. The differential equation to the elastic curve [equation (30)] is

$$\frac{d^2 y}{d x^2} = \frac{M}{E I}$$

when the curve is such that the slope $\frac{d y}{d x}$ is always small enough in comparison with unity to be neglected, or, in other words, when the inclination to the axis of the tangent to the curve, in circular measure, does not differ sensibly from the tangent of the inclination.

Next as to the sign of the left-hand side.* $\frac{d^2 y}{d x^2}$ expresses the rate of increase of the slope with respect to x, or the rate of increase of the inclination of the tangent to the axis of x. But the inclination of the tangent decreases as x increases—*i.e.*, it has a negative increase for a given positive increment of x. Hence $\frac{d^2 y}{d x^2}$ must in this case be negative. Then, writing $W(a + y)$ for the bending moment M at P, the above equation becomes—

$$\frac{d^2 y}{d x^2} = -\frac{W}{E I}(a + y)$$

* See also Appendix.

the solution of which is—

$$y + a = A \cos m x + B \sin m x^{*} \quad . \quad . \quad . \quad (63)$$

where $m^2 = \frac{W}{E I}$ and A and B are constants.

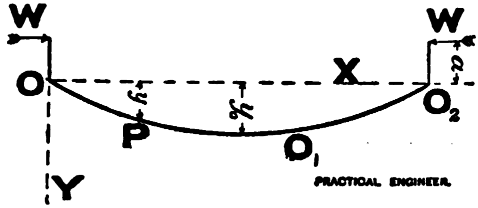


FIG. 36.

When $y = 0$, $x = 0$, and consequently $A = a$. Differentiating once, we get—

$$\frac{d y}{d x} = - a m \sin m x + B m \cos m x.$$

The left-hand side is the slope or inclination of the tangent.

$$* \quad \frac{d^2 y}{d x^2} = - m^2 (a + y)$$

Put Y for $a + y$; then

$$\frac{d Y}{d x} = \frac{d y}{d x} \text{ and } \frac{d^2 Y}{d x^2} = \frac{d^2 y}{d x^2} = - m^2 Y.$$

Multiply both sides by $2 \frac{d Y}{d x}$; then

$$2 \frac{d Y}{d x} \cdot \frac{d^2 Y}{d x^2} = - 2 m^2 Y \cdot \frac{d Y}{d x}.$$

Integrating once, we have—

$$\left(\frac{d Y}{d x}\right)^2 = - m^2 Y^2 + \text{a constant, say } m^2 b^2$$

and $$\frac{d Y}{d x} = m \sqrt{b^2 - Y^2},$$

or $$\frac{d Y}{\sqrt{b^2 - Y^2}} = m \, d x.$$

Integrating again, we get—

$$\sin^{-1} \frac{Y}{b} = m x + \text{a constant, say } e$$

then $$Y = b \sin (m x + e)$$

and $$a + y = b [\sin m x \cos e + \cos m x \sin e]$$
$$= A \cos m x + B \sin m x,$$

where $A = b \cos e$, and $B = b \sin e$.

G

to the axis of x. On account of the symmetry of the figure the tangent will be horizontal at the middle of the strut—*i.e.*, when $x = \frac{l}{2}$, l being the whole length of the strut. Therefore, in the above equation put $\frac{dy}{dx} = 0$, and $x = \frac{l}{2}$; then—

$$B = \frac{a \sin \frac{ml}{2}}{\cos \frac{ml}{2}} = a \tan \frac{ml}{2}.$$

Substituting in (63), we have

$$a + y = a \left[\cos m x + \tan \frac{ml}{2} . \sin m x \right] \quad . \quad . \quad . \quad (64)$$

The maximum deflection y_0 occurs at a point where $x = \frac{l}{2}$; then

$$a + y_0 = a \left[\cos \frac{ml}{2} + \tan \frac{ml}{2} \sin \frac{ml}{2} \right]$$

$$= \frac{a}{\cos \frac{ml}{2}} \left(\cos^2 \frac{ml}{2} + \sin^2 \frac{ml}{2} \right)$$

$$= a \sec . \frac{ml}{2} . = a \sec . \frac{l}{2} \sqrt{\frac{W}{EI}} \quad . \quad (65)$$

Should a be made equal to zero, by intention or otherwise, in this last equation, the only value of

$$\sec . \frac{l}{2} \sqrt{\frac{W}{EI}}$$

which will allow of a positive value for y_0 is infinity ; that is,

$$\frac{l}{2} \sqrt{\frac{W}{EI}} = \frac{\pi}{2}$$

or

$$W = \frac{\pi^2 EI}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

This is the result first obtained by Euler, and the value of W derived therefrom is the maximum load which it is possible for a column to withstand, under the given ideal conditions, and, *a fortiori*, under any real conditions. It will

be noticed that this expression is independent of the amount of deflection of the strut; or, in other words, given a strut originally straight, and conforming with other ideal conditions, no deflection or deformation of any kind will be noticed as the load is increased until it attains the value $\pi^2 \mathrm{E}\,\mathrm{I} \div l^2$, when the strut will be in a state of neutral equilibrium. If now the middle of the strut be displaced transversely, it will remain in the deflected position when the deflecting agent has been removed. Whether the strut is deflected or not, if any addition is made to the above critical load, the strut will collapse suddenly.

If now we return from the ideal to the real strut—*i.e.*, from equation (66) to equation (65)—we see that the deflection begins with the imposition of the first element of load, and will continue to increase with the load—though not in direct simple proportion—until rupture occurs.

By dividing both sides of the last equation by A, the area of cross-section, we get

$$f_c = \frac{\pi^2 \mathrm{E} \rho^2}{l^2} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (67)$$

Such a column as that just discussed is shown slightly bent at C D, fig. 37; the ends being round, after the nature of a ball and socket joint, the bending moment at the extremities must be zero. The column shown at E A B F is one of equal length and transverse dimensions, but the ends are *fixed*. There must be points of inflection somewhere at A and B, which points correspond to C and D in the strut with round ends, there being no bending moment at those points. If the curve E A be repeated above E, such that E H is the image of E A in C E, then H is also a point of inflection, and likewise G; and the whole curve H E A B F G is that which would be assumed by an ideal column of length G H, with ends round, and constraint being applied at the points A and B to prevent those points deviating from the straight line H A B G. From symmetry it is evident that H A = A B = B G, and therefore E F = 2 A B = H B, which is exactly the same as two round-ended columns put end to end, and constrained at the joint A from moving laterally. Now, the column H A is of equal strength with A B, and therefore the two together, end to end as above, are of the same strength as one of them alone, but the two together are equivalent to the strut E F with *fixed ends*; therefore a column with fixed ends is of the same strength as a similar column of half the length with ends

round. Let L be the length of a column, with fixed ends = $2\,l$, and after substituting in equation (66), we have

$$\left.\begin{aligned} W &= \frac{4\,\pi^2\,E\,I}{L^2} \\ \text{and} \qquad f_c &= \frac{4\,\pi^2\,E\,\rho^2}{L^2} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (68)$$

which shows that of two similar ideal columns, the one with ends fixed is four times as strong as that with round ends

The above equation for a *fixed* strut may be obtained direct from analysis, but it was thought undesirable to go through it here, as it is very similar to that in the footnote of page 81.

In the same way it may be shown that for a strut with one end fixed and the other end round, and constrained to move in the same vertical line—

$$W = \frac{2\,\pi^2\,E\,I}{l^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (69)$$

while if the round end is *unconstrained* so that it can move laterally—

$$W = \frac{\pi^2\,E\,I}{4\,l^2} \quad . \; . \; . \; . \; . \; . \quad (70)$$

These results are shown graphically in fig. 40.

Returning to equation (65) and fig. 36, the bending moment—

$$M = W\,(a + y_o) = W\,a\,\sec\frac{l}{2}\sqrt{\frac{W}{E\,I}},$$

and

$$f_b = \frac{M\,h}{I} = \frac{h\,W\,a\,\sec\frac{l}{2}\sqrt{\frac{W}{E\,I}}}{A\,\rho^2} = \frac{\frac{d}{2} \times f_c \times a\,\sec\frac{l}{2}\sqrt{\frac{W}{E\,I}}}{\rho^2}$$

but $d = n\,\rho$, and $I = A\,\rho^2$; hence—

$$f_b = \frac{n\,a\,f_c\,\sec\frac{l}{2}\sqrt{\frac{f_c}{E\,\rho^2}}}{2\,\rho} = \frac{n\,a\,f_c\,\sec\frac{l}{\rho}\sqrt{\frac{f_c}{4\,E}}}{2\,\rho} \; . \quad (71)$$

$$\text{and} \qquad f = f_c + f_b = f_c\left[1 + \frac{n\,a}{2\,\rho}\sec\frac{l}{\rho}\sqrt{\frac{f_c}{4\,E}}\right] \; . \; . \quad (72)$$

This equation permits of the maximum stress f being calculated *when the eccentricity (a) of loading is known*, or it permits *(a)* to be calculated when f is known.

As an example, take a strut 4 inches diameter and 10 feet long, made of material whose Young's modulus is 30,000,000 lbs. per sq. inch. What loads will it stand with eccentricities

of ·1 inch, 01 inch, and zero, if a maximum stress of 20,000 lbs. per sq. in. is permitted in each case?

Substituting 20,000 for f in equation 72; also 120 for l, and 1 for ρ, then finding a in terms of f_c and finally putting $\frac{W}{A} = f_c$, we have the quantities in the accompanying table.

W lbs.	150,000	160,000	170,000	180,000	190,000	200,000
a inch	·1195	·0895	·0677	·0474	·032	·0226
W lbs.	210,000	220,000	230,000	250,000		
a inch	·014	·0079	·0038	·0002		

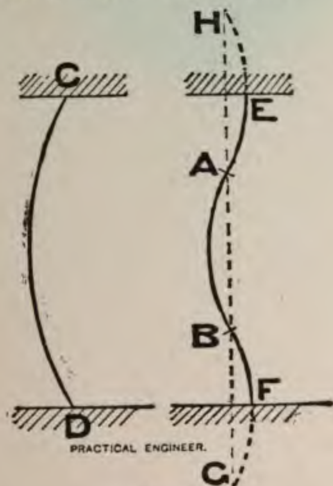
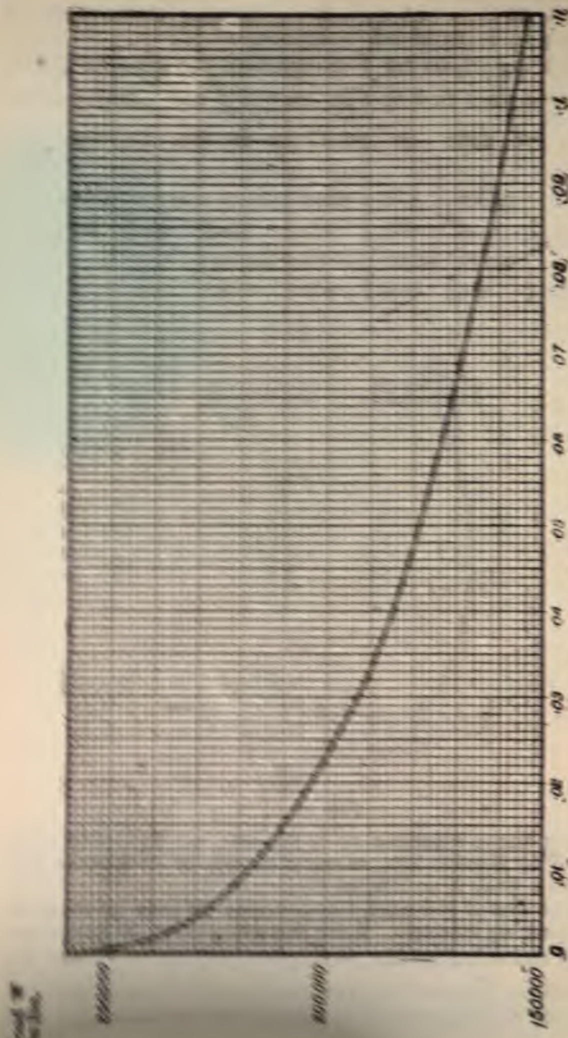


FIG. 37.

Plotting these numbers as in fig. 37A, we find that when $a = ·1$, $W = 156{,}500$ lbs., when $a = ·01$, $W = 216{,}000$ lbs., and when $a = 0$, $W = 253{,}000$ lbs.

A better idea of the nature of equation 72 may be obtained in the following manner. Take the same strut as in the last example, and calculate the direct crushing stress f_c and the bending stress f_b separately; assuming a certain eccentricity of loading, say ·1 inch and various values of W. In fig. 38, the load



ECCENTRICITY OF LOADING IN INCHES.

FIG. 37A.

W has been plotted horizontally, and the crushing stress f_c vertically from the base. This latter must be proportional to the load as the sectional area is constant, hence we get the

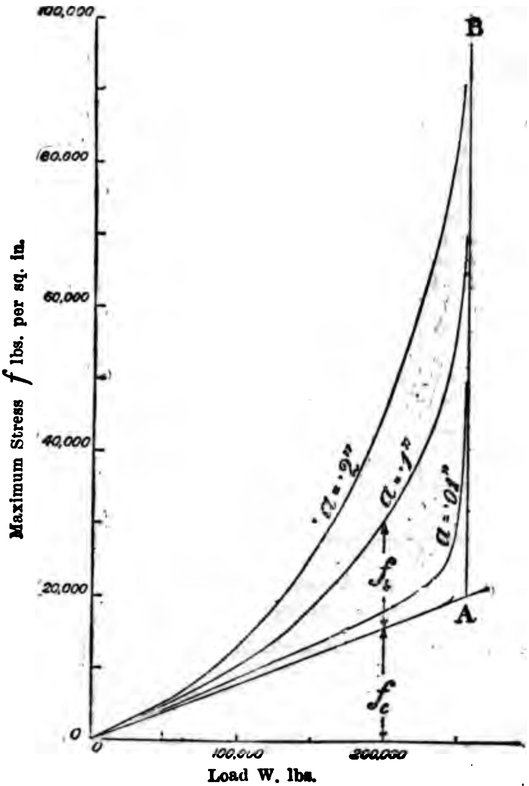


FIG. 33.

line $O\,A$ bounding the crushing stress ordinates. Now plot *above the line* $O\,A$ the bending stress caused by the various loads with a constant value of $a = \cdot 1$ inch, and we get the middle curve. The total ordinate to this curve from the base line gives the maximum stress in the material $= f_c + f_b$.

Repeat this operation for other eccentricities of loading, the curves for only two values of a being drawn. It is evident that the curves all start from the origin, and approach nearer and nearer to the asymptote $A\,B$ on the other side. Note that the bending stress with $a = \cdot 01$ inch is very small until near the point A, when it increases with enormous rapidity.

Let the eccentricity of loading become smaller still; when it is extremely small, the curve will almost consist of the two lines $O\,A$ and $A\,B$. That is to say, the bending stress is negligible until the point A is reached, when it suddenly becomes extremely great and possibly infinite, causing sudden failure of the strut. The load immediately under A is that calculated from Euler's formula (eqs. 67 or 68).

It must be noted that, with no eccentricity, the load produces only crushing stress in the material until the critical load determined by equations 67 and 68 is reached.

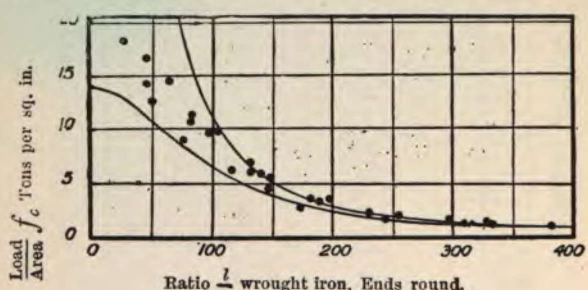


FIG. 39 —Wrought Iron, Ends Round.

For very long struts, Euler's formula gives results which agree very well with experiment, but for short struts it is very wide of the mark. This might have been expected, as short struts, like short beams, offer great resistance to bending, and long struts do not; hence the bending effect will be very small in long struts, and the stress produced will be almost entirely crushing until the critical stage is reached.

The curve in fig. 39 shows how far Euler's formula agrees with experiment, the dots representing different specimens of wrought iron that have been tested with round ends.

This enables us to calculate the constants in Gordon's formula given in the table on page 78.

For very long struts with round ends, the first term in the denominator is small compared with the second, and hence

may be neglected, Equating the load W in Gordon's formula with the same symbol in Euler's formula.

$$\frac{A f}{c\,\dfrac{l^2}{\rho^2}} = W = \frac{\pi^2 E I}{l^2} = \frac{\pi^2 E A \rho^2}{l^2}$$

After cancelling we get

$$c = \frac{f}{\pi^2 E}$$

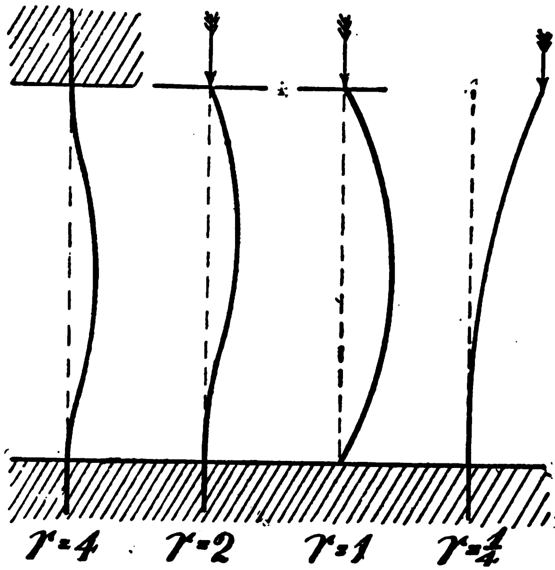


FIG. 40.

For wrought iron, $f = 14$ tons per sq. in., and $E = 12{,}000$ tons per sq. in. Substituting these we find that

$$c = \tfrac{1}{8600} \text{ for round ends.}$$

$$\text{For fixed ends } W = \frac{4\,\pi^2 E I}{l^2}$$

$$\text{Then } c = \frac{1}{4 \times 8600} = \frac{1}{34400}$$

If fact, if r is the coefficient which refers to a given method, of support (see fig. 40), then

$$c = \frac{f}{r \pi^2 E}$$

The table on page 78 gives the value of the constants to be used with Gordon's formula, and it should be understood that they have been arranged to represent the minimum strength of struts deduced from experiment; for example, in the particular case of wrought iron struts with round ends, experiment shows that the strength of the weakest struts in fig. 39 to be represented by the lower curve. Gordon's formula equation (60), when used with the constants in the table on page 78, represents this lower curve.

Gordon's formula can be most easily used in the practical design of a strut in the following manner:—

Let $W =$ total load on the strut in tons applied as nearly as possible at the centre of gravity of the ends.*

$f =$ maximum stress permitted in the material $= f_c + f_b$ on page 74.

$f_c = \frac{W}{A} =$ compressive stress, excluding that due to bending.

$A =$ sectional area of the strut in square inches.

$A_0 = \frac{W}{f} =$ a constant for any particular problem.

$c =$ the constant in Gordon's formula, page 78.

$l =$ length of strut.

and $k =$ constant in $A = k \rho^2$ (see page 152).

Now replace ρ^2 in equation 61 by $\frac{A}{k}$, and f by $\frac{W}{A_0}$ and we have

$$\frac{W}{A} = f_c = \frac{\frac{W}{A_0}}{1 + \frac{cl^2 k}{A}}$$

Re-arranging we get

$$A^2 - A A_0 = kcl^2 A_0 .$$

* When the eccentricity of loading is measureable, equation 72 or 76 must be used.

Solving this quadratic for A we get

$$A = \frac{A_0}{2} + \sqrt{\frac{A_0^2}{4} + A_0 kcl^2}$$

$$= \frac{A_0}{2}\left[1 + \sqrt{1 + \frac{4\,kcl^2}{A_0}}\right] \quad . \quad . \quad . (73)$$

Hence first find the value of A_0, then the value of c from page 78, and finally, the value of k from page 152, and substitute these quantities in equation 73. The value of f to be used in finding A_0 must, of course, be the safe working compressive stress of the material if a strut is being designed. Here it should be mentioned that the constants have been found to be used in conjunction with the *compressive* stress. If, as in the case of cast iron, the safe tensile stress has a different value, then a different value for c must be found before the formula can be made to represent failure by tension or the proper area when the tensile stress is used. Gordon's formula is really an empirical expression, and in consequence can hardly be expected to represent the individual factors that assist in destroying a strut, although it may and does represent the sum total of those factors, when there is no measurable eccentricity of loading.

In the other case, when the eccentricity is known, some equation containing the eccentricity such as 72 or 76 must be used.

As an example, let it be required to find the sectional area of a solid round steel strut, 10 feet long, with round ends, to support a load of 20 tons without any eccentricity of loading, if the maximum stress in the material be assumed to be 8 tons per sq. in.

$$\text{Now } A_0 = \frac{W}{f} = \frac{20}{8} = 2{\cdot}5 \text{ sq. in.}$$

$$l = 120 \text{ inches and } c = \tfrac{1}{7800} \text{ (from page 78).}$$

Also from page 152, $k = 4\,\pi$. Substituting these in equation 73 we have

$$A = \frac{2{\cdot}5}{2}\left[1 + \sqrt{1 + \frac{4 \times 4\,\pi \times 120 \times 120}{2{\cdot}5 \times 7800}}\right]$$

$$= 9 \text{ sq. in.}$$

$$\text{And } A = \frac{\pi}{4}\,d^2 \text{ where } d = \text{diameter} = 3{\cdot}4 \text{ inches.}$$

The solution of equation 72, when it is desired to find the sectional area, is practically impossible except by a graphical

method, and then it is somewhat tedious on account of the secant of a variable angle.

The following may lead to an easier solution of the problem relating to eccentricity of loading.

The graphical treatment given on pages 94 and 95 indicates, as mentioned in the last paragraph of page 95, that with an ideal strut having round ends, the load $P = \frac{\pi^2 E I}{l^2}$ is *practically constant for all deflections* when the deflection is not very great, or when the length of the chord of the deflection curve does not alter seriously. In this case, the bending moment $P\,d$, fig. 40A, is balanced by the moment of resistance of the material, and which must also equal $P\,d$, as long as $A\,D$ does not alter seriously.

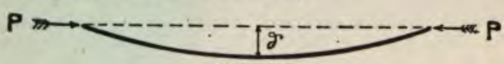


FIG. 40A.

As the cross section of the material remains the same, together with its elastic properties, this moment of resistance must equal $P\,d$ (that is, P × amount of elastic deflection), whatever be the nature of the loading.

Referring to the strut, fig. 41, whose eccentricity of loading is a,

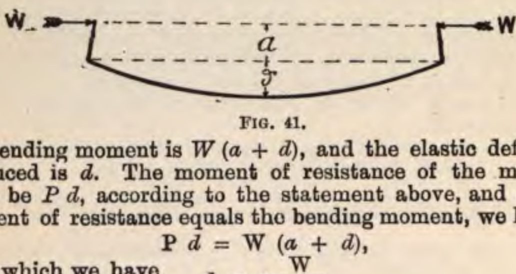


FIG. 41.

the bending moment is $W\,(a + d)$, and the elastic deflection produced is d. The moment of resistance of the material must be $P\,d$, according to the statement above, and as the moment of resistance equals the bending moment, we have—

$$\mathrm{P}\,d = \mathrm{W}\,(a + d),$$

from which we have
$$d = a\frac{\mathrm{W}}{\mathrm{P}-\mathrm{W}}$$

$$\text{and } a + d = a + a\,\frac{\mathrm{W}}{\mathrm{P}-\mathrm{W}}$$

$$= a\,\frac{\mathrm{P}}{\mathrm{P}-\mathrm{W}}$$

Also the bending moment $= \mathrm{W}\,(a + d)$

$$= \mathrm{W}\,a\,\frac{\mathrm{P}}{\mathrm{P}-\mathrm{W}}$$

Let A = sectional area of the strut,
and f_c = direct crushing stress,

$$\text{then } \frac{W}{A} = f_c$$

$$\text{Also let } f_o = \frac{P}{A}$$

$$\text{then the bending moment} = f_c \, A \, a \frac{f_o}{f_o - f_c}$$

But in equation 29, page 48, we have

$$\frac{f_b}{h} = \frac{M}{I}$$

Substituting here for M, we have

$$f_b = \frac{h}{I} + \frac{A \, a \, f_c \, f_o}{f_o - f_c}$$

$$= \frac{h \, A \, a \, f_c \, f_o}{A \rho^2 (f_o - f_c)}$$

$$= \frac{h \, a \, f_c \, f_o}{\rho^2 (f_o - f_c)}$$

The maximum stress in the material is

$$f = f_c + f_b = f_c + \frac{h \, a \, f_c \, f_o}{\rho^2 (f_o - f_c)}$$

$$= f_c \left[1 + \frac{h \, a \, f_o}{\rho^2 (f_o - f_c)} \right] \quad . \quad . \quad . \quad . \quad (74)$$

which may be written, if desirable, as

$$\left(\frac{f}{f_c} - 1\right) \left(1 - \frac{f_c}{f_o}\right) = \frac{h \, a}{\rho^2}$$

These last two equations have been derived on the assumption that f_b is taken as a compressive stress. If we take the tensile stress as we have to do in cast iron, then

$$f = f_b - f_c = f_c \left[\frac{h \, a \, f_o}{\rho^2 (f_o - f_c} - 1 \right] \quad . \quad . \quad . \quad (75)$$

$$\text{and } \left(\frac{f}{f_c} + 1\right) \left(1 - \frac{f_c}{f_o}\right) = \frac{h \, a}{\rho^2}$$

For design of a strut, we require to know the sectional area, or be able to find it in terms of the given conditions.

If we write $A = k \rho^2$, then $\rho^2 = \frac{A}{k}$

Values of k may be found in the table on page 152.

Let $h = p\rho$ where p is a constant which can generally be found from the least diameter of the strut d by the equation $h = \frac{d}{2}$ and therefore as $d = n\rho$, we find by substituting $\frac{h}{p}$ for ρ that

$$d = n\,\frac{h}{p} \text{ or}$$

$$p = \frac{n\,h}{d} = \frac{n\,d}{2\,d} = \frac{n}{2}$$

Values of n will also be found on page 152.

$$\text{Now } f_0 = \frac{P}{A} = \frac{\pi^2 E\,I}{A\,l^2} = \frac{\pi^2 E\,\rho^2}{l^2}$$

$$= q\rho^2 \text{ where } q = \frac{\pi^2 E}{l^2}$$

Also
$$f_c = \frac{W}{A} = \frac{W}{k\,\rho^2}$$

Substitute these quantities in equations 74 and 75 we get—

$$\frac{k\,\rho^4}{W} \mp \frac{1}{f}\left(\rho^2 - \frac{W}{q\,k\,\rho^2}\right) = \frac{p\,a\,\rho}{f} + \frac{1}{q} \quad . \quad . \quad . \quad (76)$$

The negative sign before the bracket being used for failure by compression, and the positive sign for failure by tension.

As an example, find the diameter of a solid round steel strut, 10 ft. long, which will just fail with a load of 20 tons, the eccentricity of loading being 2 inches.

Here $f = 18$ tons, $a = 2$, $l = 120$, $p = \frac{n}{2} = 2$, $k = 12{\cdot}5$, and $q = \frac{14000 \times 10}{14400} = 9{\cdot}7$.

Inserting these in the above equation we get—

$$\frac{12{\cdot}5\,\rho^4}{20} - \frac{1}{18}\left(\rho^2 - \frac{20}{9{\cdot}7 \times 12{\cdot}5 \times \rho^2}\right) = \frac{2 \times 2 \times \rho}{18} + \frac{1}{9{\cdot}7}$$

This can best be solved graphically; thus, after cancelling, we may write the equation as

$$\cdot 625\,\rho^4 - \frac{1}{18}\left(\rho^2 - \frac{\cdot 165}{\rho^2}\right) - \cdot 22\,\rho - \cdot 103 = 0 = y \text{ (say)}$$

Then when $\rho = 1 \quad y = \cdot256$

" $\rho = \cdot7 \quad y = -\cdot115$

" $\rho = \cdot9 \quad y = \cdot075$

" $\rho = \cdot8 \quad y = -\cdot043$

Plotting these quantities, as in fig. 41A, we find that the value of ρ, when $y = 0$, is ·85.

Then d the diameter $= 4\,\rho = 4 \times \cdot85 = 3\cdot4$ inches.

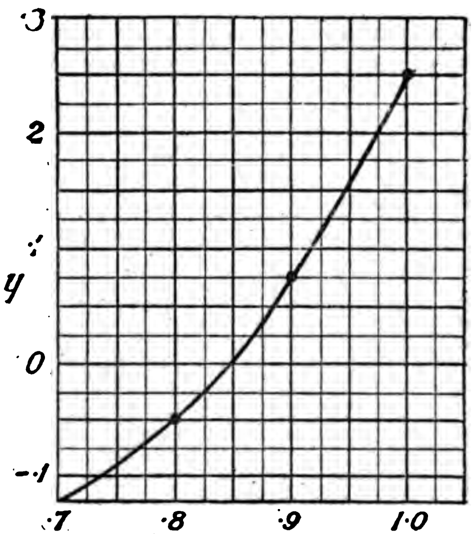


FIG. 41A.

The validity of equation 76 may be tested by comparing the results deduced from it with those deduced from equation 72. They will be found to practically coincide. And, further, the results deduced from equation 76, practically coincide with those deduced from experiment when the proper values of the physical constants have been used.

It is easily demonstrated that an initial "camber" in a strut produces a result similar to that of displacing the point of application from the centre of gravity of the end,

*Let a be the initial permanent deflection at the centre (initial camber); and let y_0 be the elastic deflection due to the load W applied at the centre of gravity of the end. Then resilient moment of resistance is the same as that of the ideal column $= P\,y_0$ where P is the load that can be applied to the ideal column.

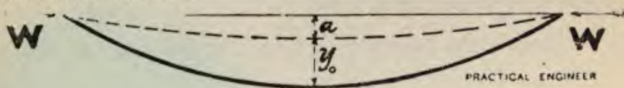


FIG 42.

The actual bending moment is $W\,(a + y_0)$ From equating the bending moment to the moment of resistance, we obtain—

$$y_0 = a\frac{P}{P - W}$$

and
$$a + y_0 = a\,\frac{W}{P - W}$$

results the same as previously obtained in the case of a displacement of the points of application of the impressed forces.

All the different causes of deviation from ideal conditions have now been treated, except that of a variation in the modulus of elasticity in different parts of the same strut. This has been most effectively done by Professor Claxton Fidler, in a paper presented to the Institution of Civil Engineers, and published in the Proceedings.

But before glancing at the results there to be found, it may be as well to see how far the graphic exposition of the laws of deflection are applicable to the long column.

The deflection curve of the ideal column is a curve of sines; hence the corresponding equation may be written in the form—

$$y = y_0 \sin a\,x$$

where y_0 is the maximum deflection, and a some constant. To determine its value, put

$$y = y_0$$

then
$$x = \frac{l}{2}$$

and from the equation, $\frac{a\,l}{2}$ must equal $\frac{\pi}{2}$. Hence

$$a = \frac{\pi}{l}$$

*See Fig. 42.

and the equation to the deflection curve is

$$y = y_0 \sin \frac{\pi}{l} . x \quad . \quad . \quad . \quad . \quad . \quad (77)$$

The bending moment being $W y$, the deflection curve is also a bending moment diagram, and consequently a stress intensity diagram; for the stress

$$= f_b = \frac{M h}{I} = \frac{W y h}{I}$$

and the area of the half-stress diagram (shaded), fig. 44,

$$= \int_0^{\frac{l}{2}} f_b . d x . = \int_0^{\frac{l}{2}} \frac{W y h}{I} . d x = \frac{W h}{I} \int_0^{\frac{l}{2}} y_0 \sin \frac{\pi}{l} x . d x.$$

$$= \frac{W h l y_0}{\pi I} = \frac{W . d . l . y_0}{2 \pi I}$$

The maximum deflection $y_0 = O K =$ the moment of the above area about O K, multiplied by $\frac{2}{d E}$ = the product of the above area, the distance of its centre of gravity from O K, and $\frac{2}{d E}$, or

$$y_0 = \frac{W d l y_0}{2 \pi I} \times K T \times \frac{2}{d E} = \frac{W l y_0}{\pi E I} \times K T.$$

Because the centre of gravity of the area (shaded) lies below the intersection of the tangents Q T, O T, at the extremities of that part of the diagram under consideration But—

$$\frac{O K}{K T} = \tan O T K = \frac{d y}{d x} \text{ when } x = O$$

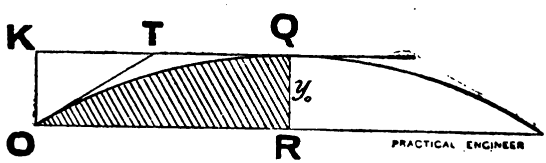


Fig. 44.

and $$\frac{d y}{d x} = a y_0 \cos a x = \frac{\pi}{l} y_0 \text{ when } x = O$$

hence $$\frac{\pi}{l} y_0 = \frac{y_0}{KT} \text{ and } K T = \frac{l}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

Substituting this value for K T in above, we obtain—

$$W = \frac{\pi^2 E I}{l^2}$$

the same as previously obtained.

If it is attempted to solve this problem graphically, direct, it will be found that it cannot be done in the ordinary way; but if a certain deflection be assumed, the resilient load W can be obtained. This being done for a series of deflections of the same column, it will be found that the corresponding derived values of W will be as nearly as possible the same.

CHAPTER IX.

Effect of Variation of Elasticity on the Strength of a Strut.

We may now pass on to the consideration of the effect of a difference of modulus of elasticity in two different portions of the same strut, and, for the sake of simplicity, we will assume that all of the material on one side of the neutral surface will have one modulus of elasticity, while all the material on the other side will have some other modulus, corresponding to the two flanges. Using the same notation as before, the maximum bending moment

$$= W\,y_o = f_b\,\frac{I}{h}$$

therefore $$f_b = \frac{W\,h\,y_o}{I} = \frac{W\,h\,y_o}{A\,\rho^2} = f_c\,.\,\frac{h\,y_o}{\rho^2}.$$

This is the maximum stress due to bending only that can occur in the strut. The stress at any particular section is given by the length of an ordinate to the curve O Q, fig. 44; and the average stress intensity f_a over the whole length of one flange can be obtained from the equation—

$$f_a \times \frac{l}{2} = \text{shaded area O Q R.}$$

When Q R represents the maximum bending stress f_b, the

$$\text{area O Q R} = \int_o^{\frac{l}{2}} y\,d\,x = \int_o^{\frac{l}{2}} y_o \sin\frac{\pi}{l}\,x\,.\,d\,x.$$

$$= y_o\,\frac{l}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

and $$y_o = \text{Q R} = f_b$$

therefore $$f_a = \frac{2\,f_b}{\pi} = \frac{2}{\pi}\,.\,f_c\,.\,\frac{h\,y_o}{\rho^2}.$$

Average total compressive stress throughout the concave flange

$$= f_a + f_c = f_c\left(1 + \frac{2\,y_o\,h}{\pi\,\rho^2}\right)$$

Average compressive stress in convex flange

$$= f_c\left(1 - \frac{2\,y_o\,h}{\pi\,\rho^2}\right)$$

Amount of compression in half length of concave flange

$$= \text{average strain} \times \text{original length}$$

$$= \frac{\text{average stress}}{E_1} \times \text{original length}$$

$$= \frac{l}{2\,E_1} \cdot f_c\left(1 + \frac{2\,y_o\,h}{\pi\,\rho^2}\right)$$

the suffix 1 denoting the concave flange, and the suffix 2 denoting the convex flange; the two moduli by hypothesis being different.

Amount of compression in half length of convex flange, in the same way,

$$= \frac{l}{2\,E_2} \cdot f_c \cdot \left(1 - \frac{2\,y_o\,h}{\pi\,\rho^2}\right)$$

The difference in length of the two half flanges = compression of concave flange − compression of convex flange,

$$= \frac{l}{2\,E_1} \cdot f_c \cdot \left(1 + \frac{2\,y_o\,h}{\pi\,\rho^2}\right) - \frac{l}{2\,E_2} f_c\left(1 + \frac{2\,y_o\,h}{\pi\,\rho^2}\right)$$

$$= \frac{l}{2\,E_1\,E_2} \cdot f_c \cdot \left[E_2 - E_1 + \frac{2\,y_o\,h}{\pi\,\rho^2}(E_1 + E_2)\right]$$

But the difference in length of the flange is represented in fig. 45 (see page 102) by S O, and

$$\tan \text{O R S} = \frac{\text{O S}}{\text{S R}} = \frac{\text{difference of length of flanges}}{\text{depth of strut}}$$

The figure has been much exaggerated so as to show the several quantities distinctly; were it drawn to scale, R S would be as nearly as possible the least transverse dimension d of the strut, and the strut being approximately straight, the difference between the lengths of the flanges would be sensibly represented by S O.

The tangent T O is drawn at O, then, because R O is normal to O T, and O K is perpendicular to K T. The two

triangles O R S and O T K are similar, and hence the angle O S R = the angle O T K; therefore

$$y_o = \mathrm{OK} = \mathrm{TK} \tan \mathrm{OTK} = \mathrm{TK} \tan \mathrm{ORS}$$

$$= \mathrm{TK} \times \frac{\text{difference of length of flanges}}{\text{depth of strut}}$$

$$= \mathrm{TK} \times \frac{l}{2\,d\,\mathrm{E}_1\,\mathrm{E}_2} \cdot f_c \cdot \left[\mathrm{E}_2 - \mathrm{E}_1 + \frac{2\,y_o\,h}{\pi\,\rho^2}(\mathrm{E}_2 + \mathrm{E}_1)\right]$$

From equation (78),

$$\mathrm{KT} = \frac{l}{\pi}$$

and E, the average value of the modulus, may be substituted for

$$\frac{\mathrm{E}_1 + \mathrm{E}_2}{2}$$

also $\mathrm{E}_1\,\mathrm{E}_2$ does not differ much from E^2. After making these substitutions, we have—

$$y_o = \frac{l^2}{\pi\,d\,\mathrm{E}} \cdot f_c \left[\frac{\mathrm{E}_2 - \mathrm{E}_1}{\mathrm{E}_2 + \mathrm{E}_1} + \frac{2\,h\,y_o}{\pi\,\rho^2}\right]$$

Solving this equation for y_o, we get

$$y_o = \frac{\dfrac{\pi\,\rho^2}{d} \cdot f_c}{\dfrac{\pi^2\,\rho^2\,\mathrm{E}}{l^2} - f_c}\left[\frac{\mathrm{E}_2 - \mathrm{E}_1}{\mathrm{E}_2 + \mathrm{E}_1}\right]$$

If we call the direct crushing stress of the ideal column f_i, the above may be written—

$$y_o = \frac{f_c}{f_i - f_c} \cdot \frac{\pi\,\rho^2}{d} \cdot \left[\frac{\mathrm{E}_2 - \mathrm{E}_1}{\mathrm{E}_2 + \mathrm{E}_1}\right]. \quad . \quad . \quad . \quad . \quad (80)$$

Should the strut be composed of two flanges with counter-bracing, then

$$\rho = \frac{d}{2}$$

approximately. It is in this form that it is given by Professor Fidler in the Minutes of Proceedings of the

Institute of Civil Engineers, vol. lxxxvi., page 291. Now—

$$f_b = \frac{M}{I} h = \frac{W y_0 d}{2 A \rho^2} = f_c . \frac{d}{2 \rho^2} . \frac{f_c}{f_i - f_c} . \frac{\pi \rho^2}{d} \left[\frac{E_2 - E_1}{E_2 + E_1} \right]$$

$$= \frac{\pi}{2} . \frac{f_c^2}{f_i - f_c} \left[\frac{E_2 - E_1}{E_2 + E_1} \right]$$

and, consequently,

$$f = f_c + f_b = f_c \left[1 + \frac{f_c}{f_i - f_c} . \frac{\pi}{2} \left(\frac{E_2 - E_1}{E_2 + E_1} \right) \right]$$

also,

$$\frac{\pi^2 E \rho^2}{l^2} = f_i = \frac{f_c}{f - f_c} . \left[f - f_c \left(1 - \frac{\pi}{2} \frac{E_2 - E_1}{E_2 + E_1} \right) \right]$$

therefore

$$\frac{l^2}{\rho^2} = \frac{\pi^2 E (f - f_c)}{f_c \left[f - f_c \left(1 - \frac{\pi}{2} . \frac{E_2 - E_1}{E_2 + E_1} \right) \right]} \quad . \quad . \quad . \quad . \quad (81)$$

This last equation and the third and fourth above do not coincide with Professor Fidler's in every symbol, the difference being that he has substituted 2ρ for d in the denominator, and retained h instead of $\frac{d}{2}$ in the numerator of the fourth equation above. He also estimates that the probable limiting value of

$$\frac{E_2 - E_1}{E_2 + E_1} \text{ is about } \tfrac{1}{3},$$

and that of the coefficient—

$$\frac{\pi}{2} \frac{h}{\rho} . \frac{E_2 - E_1}{E_2 + E_1} \text{ about } \cdot 4.$$

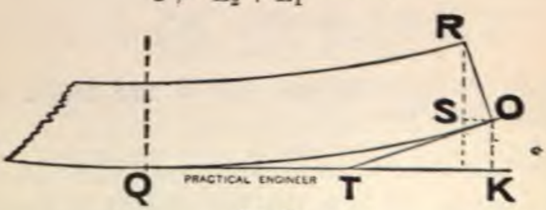


FIG. 45.

SUMMARY OF THE THEORY OF STRUTS.

In designing an end loaded strut, the one feature which decides the equation to be used is the eccentricity of loading.

If this should be zero, then Gordon's formula should be used. It can be transformed into the following (see page 91).

$$A = \frac{A_0}{2}\left[1 + \sqrt{1 + \frac{4\,k\,c\,l^2}{A_0}}\right]$$

where

A = required sectional area of the strut.

$A_0 = \frac{W}{f} = \frac{\text{load}}{\text{maximum stress}}$ (see page 90).

$k = \frac{A}{\rho^2}$ (see column 5, page 152). Also page 107.

ρ = least radius of gyration.

c = constant (see page 78).

l = length of strut.

For the construction of this formula, see pages 73 to 79 and page 90.

Where it is known beforehand that the ratio length to radius of gyration is very large, Euler's formula can be used. It is

$$W = \frac{r\,\pi^2\,E\,I}{l^2}$$

Where W = load on end of strut.

E = Young's Modulus of Elasticity. } (see table,
r = constant. } page 78).

I = Moment of Inertia of Cross Section (see table, page 152).

The construction of this formula will be found on pages 80 to 84.

Should the eccentricity of loading be not zero and measurable, then it is generally easiest to use the following equation—

$$\frac{k\,\rho^4}{W} + \frac{1}{f}\left(\rho^2 - \frac{W}{q\,k\,\rho^2}\right) = \frac{p\,a\,\rho}{f} + \frac{1}{q}$$

the negative sign before the second term being used when f is compressive, and the positive sign when f is tensile.

W = load on end of strut.

a = eccentricity of loading.

= distance between the point of application of the load and the axis of the strut.

ρ = radius of gyration of cross-section.

$k = \frac{A}{\rho^2}$ (see table, page 152 and page 107).

l = length of strut.

f = maximum stress permitted in the material (see pages 74, 84, or chapters xi. and xii.).

$= f_b + f_c$ if compressive.

$= f_b - f_c$ if tensile.

$q = \frac{r \pi^2 E}{l^2}$ (see page 94).

r = constant (see fig. 40 and table, page 79).

$p = \frac{h}{\rho}$ from $h = p\rho$ (see page 94).

h = least distance from axis of strut to the outer skin (see equation 29 and page 46).

$= \frac{d}{2}$ for symmetrical sections.

Other forms of the general equation given above are indicated on page 94.

It is also possible to use equation 72, page 84, but it is of a very inconvenient form, and is not to be compared with that given above for general use.

Professor Fidler's equation (81), or 4 lines above it on page 102, may also be used. It is very similar to equation 76, and represents the strength of struts very well indeed. The reader should consult Professor Fidler's treatise on Bridge Construction for further information concerning it.

The following table of sections represents the dimensions of rolled steel joist, made by Messrs. Skelton & Co., of London, with specially *wide flanges*, so that they will be most economical for strut purposes.

Dimensions.		Sectional Area.	Moments of Inertia.		Moments of Resistance (Modulus of Section).	Least Radius of Gyration.
Height.	Width	sq. in.	Axis X X.	Axis Y Y.	Axis X X.	Axis Y Y.
ins.	ins.					
8⅝	8⅝	12·8	177	53	41	2·04
9½	9½	15·0	246	73	52	2·21
9⅞	9⅞	16·3	290	86	59	2·30
10¼	10¼	17·9	344	102	67	2·39
10⅝	10⅝	19·1	397	118	75	2·49
11	11⅞	20·4	457	136	83	2·58
11⅜	11⅜	21·9	525	154	92	2·78
11⅞	11⅞	23·6	605	180	102	2·76
12⅜	11⅞	24·9	723	189	115	·275
13⅜	11⅞	25·9	846	194	126	2·74
14⅛	11⅞	28·2	1019	211	144	2·74
15	11⅞	29·6	1188	220	159	2·73
15¾	11⅞	31·6	1388	233	176	2·72
16¾	11⅞	33·2	1638	242	196	2·70
17¾	11⅞	35·6	1941	256	219	2·68
18¾	11⅞	37·5	2275	267	243	2·67
19⅝	11⅞	40·6	2671	281	271	2·63
21⅝	11⅞	44·6	3502	302	324	2·60
23⅝	11⅞	46·6	4308	304	365	2·56
25⅝	11⅞	48·8	5223	307	408	2·51
29½	11⅞	52·0	7270	308	492	2·43

The numerical value of k is not constant for all sizes of the same form of cross-section, as will be seen from the annexed table of dimensions of sections of rolled joists. The table has been divided into two parts, the first

containing sections which are fairly regular—*i.e.*, which have a fairly uniform value for k; and the second containing other sections which are generally manufactured, but whose value for k is not such as can be conveniently applied in the above equation. A large value of k indicates a large area of cross-section for a given radius of gyration, and consequently the strut will not be so efficient as one with a smaller value of k. This indicates, too, that sections with a smaller value of k than the average may be used in the

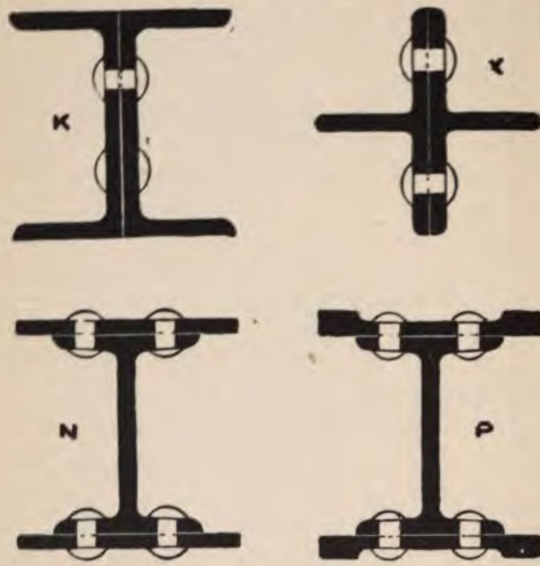


FIG. 47.

preceding equation without endangering the safety of the strut. The average value of k is 7·5 for rolled joists, as found in the first portion of the following table; and the maximum deviation in its value is about 20 per cent on either side of the average. This maximum deviation produces a probable error of about 2 per cent in the area of cross-section.

The dimensions in the following tables have been chiefly ·ined from particulars and tables kindly furnished by

ROLLED IRON AND STEEL JOISTS.

TABLE I.—Regular sections.

Transverse dimensions in inches.		Area of cross-section in square inches.	Weight of a piece of joist 1 ft. long in pounds.	k
3	by 1½	1·2	4	8·5
4	1¾	1·64	8·5	8·7
4	3	3·72	12·75	6·7
4½	3	4·1	14	7·3
5	3	4·44	15·25	8
6	3	4·7	16	8·4
6¼	3½	3·85	13	6·5
6¼	3½	5·25	18	7
7	3¾	5·85	20	6·7
8	4	7·32	25	7·4
9	3¾	7·1	24·25	8
9¾	4½	10·8	37	8·6
10	4½	9·2	31·5	7·3
10	5	10·2	35	6·6
12	5	12·6	43	8·3
12	6	15·8	54	7·1
12	6½	19·18	64	7·3
13	5	11·7	40	7·5
13¾	5½	14	47	7·4
14	6	16·6	57	7·4
15	5	12·2	42	7·9
15¾	6¼	18·59	62	8
15	6	17·8	61	8
16	5	14·6	50	9·3
16	6	18·8	61·5	8·4
17¾	6¾	23·3	78	8·2
18	7	21·8	75	7·2
19¾	7¼	28·2	94	8·7

TABLE II.—Irregular sections.

Transverse dimensions in inches.		Area of cross-section in square inches.	Weight of a piece of joist 1 ft. long in pounds.	k
3	by 1¼	1·46	5	15
3	3	2·55	8·5	4·5
3	3	3	10·25	5·3
3½	1½	1·74	6	12·4
3½	3	3·14	10·75	5·6
4¾	1¾	1·94	6·5	10
4¾	1¾	2·7	9·25	14·1
4	3	2·83	9·5	5
5	3	3·05	10·2	5·5
5	4½	5·4	18	4·2
5	4½	6·9	23·75	5·5
5	5	7·46	25·5	4·8
5¼	1½	2·62	9	18·6
5½	2	3·2	11	12·8
6	2	3·72	12·75	14·8
6¼	2	2·84	9·5	11·4
6	4½	5·55	19	4·4
6	5	7·6	26	6
7	2¼	3·3	11	10·5
7	3¾	4·95	16·5	5·5
8	4	5·55	18	5·5
8	5	9·15	31·25	5·9
8	6	10·55	36	4·7
9¼	4	6·13	20·5	6·2
9	7	16·9	58	5·6
10	5	9	29	5·8
10	6	13·3	45·5	6
12	5	9·4	32	6·1
12	6	13·2	44	5·9
14	6	13·2	45	6

Messrs. Dorman, Long, and Co., of Middlesbrough and London.

In concluding this discussion on the strength of struts, a few sections of the forms in general use are here appended; but no attempt has been made to give any of the constant quantities relating thereto on account of the great diversity of relative dimensions. Section K, fig. 47, is composed of a pair of channel irons riveted back to back, with the rivets arranged zigzag. For small sections, a single row of rivets

Dimensions of Tee Iron and Angle Iron in General Use.

Tee Irons.				Angle Irons.			
Transverse dimensions.		Thickness.		Transverse dimensions.		Thickness.	
6 by	3	3/8 and	1/2	8 by	8	1/2 to	1
5	3	3/8	1/2	7	7	1/2	1
5	2½	3/8		6	6	1/2	1
4	5	1/2		5	5	7/16	7/8
4	4	1/2		4½	4½	3/8	7/8
4	3½	3/8	1/2	4	4	5/16	11/16
4	3	3/8	1/2	3½	3½	5/16	3/4
4½	3½	7/16		3	3	1/4	11/16
3½	3½	3/8	1/2	2¾	2¾	1/4	5/8
3½	3	3/8	1/2	2½	2½	1/4	5/8
3	3	3/8	1/2	2¼	2¼	1/4	1/2
3	2½	3/8	1/2	2	2	3/16	3/8
2½	2½	1/4	1/2	1¾	1¾	3/16	3/8
2¼	2¼	1/4		1½	1½	1/8	5/16
2	2	1/4	3/8	1¼	1¼	1/8	5/16
2	1½	1/4	3/8	1	1	3/32	1/4
1½	2	1/4	5/16				
1¾	1¾	5/16	3/16				
1½	1½	1/4					

Bulb angles.		Round-backed angles.	
8 by 3		4 by 4	$\frac{3}{8}$ to $\frac{3}{4}$
$7\frac{1}{2}$ 3		4 3	$\frac{5}{16}$ $\frac{3}{4}$
7 3	$\frac{7}{16}$ to $\frac{11}{16}$	$3\frac{1}{2}$ $3\frac{1}{2}$	$\frac{3}{8}$ $\frac{3}{4}$
$6\frac{1}{2}$ 3		3 3	$\frac{5}{16}$ $\frac{5}{8}$
6 $3\frac{1}{2}$		$2\frac{3}{4}$ $2\frac{3}{4}$	$\frac{5}{16}$ $\frac{9}{16}$
6 3	$\frac{3}{8}$ to $\frac{5}{8}$	$2\frac{1}{2}$ $2\frac{1}{2}$	$\frac{1}{2}$ $\frac{5}{8}$
$5\frac{1}{2}$ 3			

Channel Iron.

12 by	$3\frac{1}{2}$ by	$\frac{7}{16}$ to	$\frac{5}{8}$		6 by	$3\frac{1}{8}$ by	$\frac{1}{2}$ to	$\frac{5}{8}$
10	4	$\frac{7}{16}$			6	$2\frac{7}{8}$	$\frac{7}{16}$	$\frac{9}{16}$
10	3	$\frac{3}{8}$			6	$2\frac{1}{2}$	$\frac{7}{16}$	$\frac{1}{2}$
9	3	$\frac{3}{8}$			$5\frac{1}{8}$	$2\frac{7}{8}$	$\frac{1}{2}$	$\frac{9}{16}$
8	$3\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$		$4\frac{1}{2}$	2	$\frac{3}{8}$	$\frac{9}{16}$
$7\frac{1}{8}$	$3\frac{3}{4}$	$\frac{7}{16}$	$\frac{5}{8}$		4	3	$\frac{1}{2}$	
7	$3\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$		$3\frac{1}{2}$	$1\frac{1}{2}$	$\frac{3}{8}$	
7	3	$\frac{1}{2}$						

is quite sufficient. Section X is formed of a pair of tee irons riveted back to back, but the back is twice as wide as the leg is long. The third section N in the same figure is of the ordinary plate-girder form, the central connection being made of a double tee rolled joist; while the last section P is a special form of the American Bridge Company's manufacture. Professor Fiddler gives 2·4 as the average value of k for this latter section. The object is to make the radius of gyration as great as possible with the plate-girder form of construction. Fig. 48 shows two forms of the box type, of which the section Q is made up of a pair of channel irons connected together by a pair of plates. In section S the channel irons are replaced by a couple of web plates and four angle irons. The section fig. 49 is generally used where flexure will probably take place in a vertical direction when the strut is situated as shown. The central double tee piece stiffens the strut generally, and especially the two

boom plates. It is also designed (as most struts ought) to be riveted up entirely by hydraulic machinery. The strut shown at C, fig. 50, is known as the Phœnix column. It is of American origin, and shows great strength in comparison

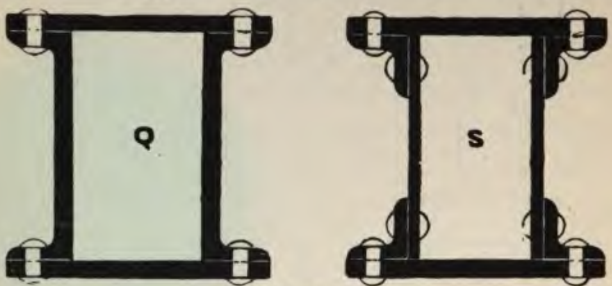


FIG. 48.

with transverse dimensions. It is formed of a number of sections of mild steel or wrought iron riveted together, the number being either four, six, or eight according to the size of strut. In the figure the flanges are much larger in proportion to the annular body than is found in practice.

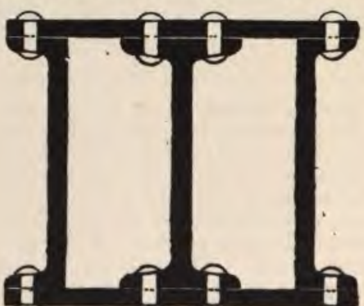


FIG. 49.

A similar sort of strut, section D, is made from a set of Lindsay's troughs riveted together. These troughs are largely used in the construction of the floors of railway bridges. In fig. 51 is shown an economical form of strut, consisting of a pair of channel irons connected with diagonal

bracing. It is to be very frequently met with in bridge work or other structures much exposed to atmospheric

FIG. 50.

influences. The open work allows of a coat of paint being easily applied to both the inside and out.

CHAPTER X.

EXAMPLES OF STRUTS AND COLUMNS.

EXAMPLE 1.—A movable gantry crane in a machine or erecting shop runs upon a race supported by a number of hollow round cast-iron pillars. The race consists of a pair of rails placed upon two continuous girders, to which the columns are attached by bolts, the two ends of each column terminating in flanges. The weight of race supported by each column is 2 tons, the weight of gantry is 6 tons, and the maximum weight to be lifted by the gantry is 30 tons. The length of each column is 20 ft. What is the diameter of the column?

In hollow columns, it is usual to make the thickness of metal about one-tenth of the outer diameter. Let D be the outer diameter, and d the inner diameter. The sectional area of metal is then

$$\frac{\pi}{4}(D^2 - d^2) \text{ square inches.}$$

Putting in $\cdot 8\,D$ for d, the area is given by the expression $\cdot 09\,\pi\,D^2$. Using the previous notation where ρ is the radius of gyration of the section,

$$n\rho = D.$$

The previous table of values of n gives 3·1 as its value for hollow cylinders whose thickness of metal is one-tenth of the outer diameter.

Also A, the area of cross-section, $= k\rho^2$; therefore

$$k = \frac{A}{\rho^2} = \frac{\cdot 09\,\pi\,D^2}{\frac{D^2}{n^2}} = \cdot 09\,\pi\,n^2 = 2\cdot 7.$$

Then, assuming 6 tons per square inch as the working stress of cast iron, and obtaining the value of the constants from page 78, the several quantities may be substituted in equation (76).

The load W = half the lifted weight + half the weight of gantry + the portion of race supported by each pillar $= 15 + 3 + 2 = 20$ tons. The sectional area is measured in square inches; hence the length of pillar must be measured in inches. Each pillar is rigidly attached to the continuous girder at its upper end, and to the foundation stone at its lower end. At the same time it is generally attached (in an erecting shop) at its upper end to the pillar supporting the roof, so that it is very similar to a strut *fixed* at both ends; but as this cannot be relied upon, we may assume one end to be fixed. Then we find after substituting in the above-mentioned equation that

$A = 13$ sq. ins., and $D = \sqrt{\dfrac{A}{\cdot 09\,\pi^2}} = 7''$ nearly, and thickness $= \frac{7}{10} = \frac{3}{4}''$ nearly.

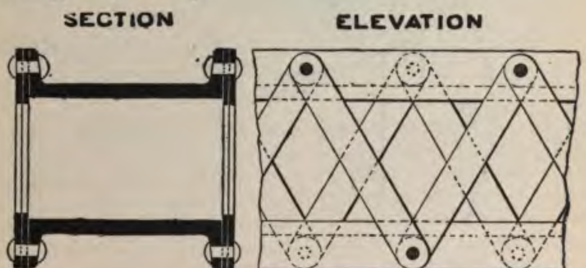


FIG. 51.

The above value of D is that at its middle section.

Pillars are sometimes made slightly taper for the sake of appearance. If this is done, the diameter at the bottom must be increased slightly, while that at the top must be slightly decreased, leaving the diameter in the middle the same as obtained above.

EXAMPLE 2.—A water tank is 5 ft. deep, 10 ft. wide, and 20 ft. long. It is supported by six cast-iron columns, each 30 ft. long (three on each side). The weight of metal in the tank and bearing girders is 7 tons. The tank rests upon a number of cross girders, which are supported by a pair of girders whose length is the same as that of the tank. What is the diameter of the columns?

The columns are assumed to have flanges at their extremities, which are secured to their attachments by bolts. The columns would probably give way by a bending over of

their upper extremities, accompanied by a horizontal displacement, as shown in fig. 52 at A B. At the extremities the axis must be vertical, because the flanges are securely bolted to the foundation in the one case and the longitudinal girder in the other. It is evident from the figure that the pillar A B is half the pillar A C, which represents one with ends *fixed* and maintained in the same vertical line. The result is that a column such as A B will bear the same load as another column A C of double the length, with its ends fixed. But a column which will bear the same load as another twice its length, with ends fixed, is a column with round ends (see fig. 33); hence the column A B is of the same strength as a column of the same length with its ends round.

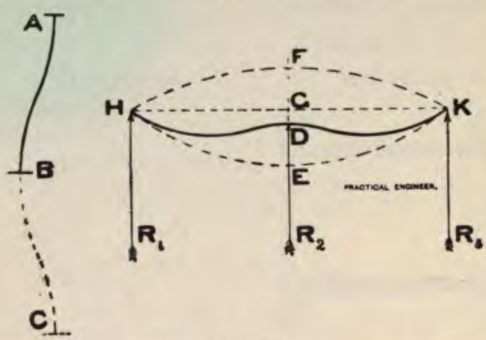


FIG. 52.

The thickness of metal will be assumed to be one-tenth of the outer diameter, as in the previous example, and the constants taken from the tables, page 78, while 6 tons per square inch may be taken as the working stress.

The value of k is 2·7, as in last example, when the thickness is one-tenth the outer diameter, and W is the load on one pillar. The weight of water contained in the tank when full

$$= \frac{20 \times 10 \times 5 \times 62\frac{1}{2}}{2240} \text{ tons} = 28 \text{ tons}.$$

Total load uniformly distributed over the two longitudinal girders $= 28 + 7 = 35$ tons, or $17\frac{1}{2}$ tons on each girder. Now, if the upper ends of the girders are maintained at the same level, and the girder assumed to be straight and of uniform section, as a rolled joist would be, then the amount of load sustained by each of the outside pillars is three-

sixteenths of the load upon the girder,* the intermediate column supporting five-eighths of the girder load. The load on each of the outside pillars would then be about $3\frac{1}{2}$ tons, while that on the centre pillar would be a little under 11 tons. It would be an exceedingly difficult practical task to

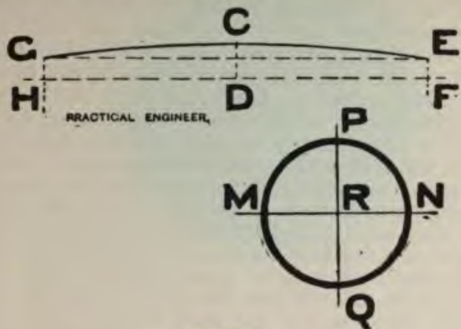


FIG. 53.

so align the columns that each would just sustain the load allotted to it, and at the same time it would be equally as difficult to arrange them so that they should not sustain *more* than their allotted load; for, should the centre column project, however slightly, above the line joining the three

* The deflection at the centre of a girder, supported at each end and uniformly loaded, is given in equation (33) as

$$\frac{5}{384}\frac{W l^3}{E I}.$$

The deflection of the same girder due to an upward concentrated load P at its centre is, by equation (32),

$$-\frac{8}{384}\frac{P l^3}{E I}.$$

Therefore, when both loads are on the girder at the same time, the deflection is the sum of the deflections due to the individual loads; or,

$$= \frac{5}{384}\frac{W l^3}{E I} - \frac{8}{384}\frac{P l^3}{E I}.$$

But if the three props are on the same level, then this deflection is zero, and hence—

$$5 W - 8 P = 0,$$

or, $\qquad$ P, the central load, $= \frac{5}{8}$ W,

and, consequently, each of the outer props support three-sixteenths of the load W.

upper extremities, it would sustain more than five-eighths of the girder load, and, should it not quite reach that altitude, it would not support so much as five-eighths. Again, should the columns be properly aligned, and the bearing surface of the girder be not quite plane, the figures given as representing the fraction of load sustained by each pillar will be inaccurate.

It is, therefore, quite necessary to design the centre column to sustain at least five-eighths of the girder load. Now, in many cases it is cheaper to make the end pillars of the same dimensions as the middle one, although by so doing they will be inordinately strong; but the amount of load supported by each will be thereby more nearly equalised. This is easily seen, thus: If the conditions are the ideal ones suggested above, the area of cross-section is roughly proportional to the load supported, other circumstances being the same; and, as the lengths are equal throughout, each of the (elastic) pillars will be compressed through the same amount—or, in other words, the total strain of each pillar will be the same, and if the ends were in a straight line before the load were applied, they would remain in a straight line after the load were applied, and, therefore, the imposition of the load would not in any way vitiate the conditions of the problem. But if all three pillars were of the same dimensions as the middle one, and perfectly aligned before the imposition of the load, the smaller load upon the greater area of the outer pillars would produce in them a total strain less than that of the outer pillars in the previous case of different dimensions; and the final level of the outer pillars being then above that of the middle one, the former would sustain more of the whole load than when the centre pillar (of greater dimensions than the outer pillars) was in perfect alignment after the imposition of the load. This at once suggests the following problem: Given that all the pillars are of the same dimensions, and sustain the same fraction of the load, What must be the difference between the altitude of the outer and middle pillars? As the load is the same on each, and as all are of the same dimensions, each will be strained through the same distance, and the relative heights of the ends will remain the same both after and before the imposition of the load, and, consequently, the conditions will remain constant.

Let R_1, R_2, and R_3 be the three equal reactions (fig. 52), and let P be the total load on the three columns; then, if the centre column be supposed to be removed, H E K will be

the deflection curve, and the maximum deflection at the centre will be C E, given by the expression—

$$C\,E = \frac{5}{384}\,\frac{P\,l^3}{E\,I},$$

the load P being uniformly distributed [see equation (33)].

Now assume the load P is removed, and a single upward force applied at the centre of magnitude equal to R_2. The corresponding deflection at the centre C F is given by

$$C\,F = \frac{8}{384}\,\frac{R_2\,l_3}{E\,I}. \quad \text{[See equation (32).]}$$

If now the downward distributed load P and the upward concentrated load R_2 are both applied simultaneously, the resulting deflection must be the sum of the deflections due to both loads separately; and remembering that the loads act in opposite directions, and consequently the deflections occur in opposite directions, the sum of the deflections must be C E + (– C F), which is C D in the figure. Hence,

$$C\,D = C\,E - C\,F = \frac{5}{384}\,\frac{P\,l^3}{E\,I} - \frac{8}{384}\,\frac{R_2\,l^3}{E\,I}.$$

$$\text{But } R_1 = R_2 = R_3 \text{ and } P = R_1 + R_2 + R_3\,;$$

then, substituting for R_2 in the above equation,

$$C\,D = \frac{P\,l^3}{384\,E\,I}\left(5 - \frac{8}{3}\right) = \frac{7}{1152}\,\frac{P\,l^3}{E\,I},$$

that is, the upper end of the middle pillar requires to be set a distance equal to

$$\frac{7}{1152}\,\frac{P\,l^3}{E\,I}$$

below the level of the two end pillars, so that the load sustained by each pillar will be the same. Of course, it is as difficult to exactly locate the central pillar end, wherever it is suggested to fix it; and, therefore, it is just as difficult to so arrange the pillars that they all sustain the same load, as it is to fix them so that the centre one sustains five-eighths of the girder load. In each of the cases considered the longitudinal girder has been assumed to be *supported* by the end pillars, and not rigidly attached to them, as the problem suggests. If we go a step further, and rigidly attach the girder to the outer posts, then it is clear that any deflection of the girder also bends the outer posts. The resistance of the outer posts to bending will relieve the

centre post of some of its load, and, in consequence, the three loads upon the pillars become still more nearly equalised. The exact solution of the problem, taking everything into consideration, is extremely long and tedious, and altogether out of the province of such a work as the present one. The above cases have been introduced for the sake of showing how different conditions affect the final result.

It is general in practice to make all the pillars of the same dimensions, and to arrange them as nearly as possible on the same level; it is then pretty certain that the centre column does not support more than five-eighths of the total girder load, and consequently the outer columns also will not support more than five-eighths of the load.

Maximum load on any column equals five-eighths of $17\frac{1}{2}$ tons, say 11 tons $=$ W in equation (73). Then, after substituting, we obtain $A = 18{\cdot}8$ square inches.

But $A = {\cdot}09\ \pi\ D^2$, and therefore $D = 8\frac{1}{2}$ in.

With long columns, such as those in the above problem, it is customary to brace them together at their centres.

Example 3.—A maximum load of 2 tons is to be lifted by a hydraulic lift through a maximum distance of 15 ft. The weight of cradle and ram is, say, not more than $1\frac{1}{2}$ ton. What is the diameter of the solid wrought-iron ram?

The cradle slides loosely between guides, so that the upper end of the ram is guided in a vertical line, but is *not fixed*. The lower end passes through a stuffing box, which is not sufficiently stiff to constitute a *fixed* support. We may, therefore, consider the ram to have round ends.

Here it will be advisable to limit the working stress to 5 tons per square inch. Inserting the several values in equation (73) we obtain:

$$A = 4 \text{ square inches,}$$

and, consequently, $d = 2{\cdot}3$ in.

Example 4.—The jib of a crane is required to sustain a direct thrust of 13 tons. The material is pine, and the length from centre to centre of pins is 18 ft. What is the diameter of the jib at the centre?

The safe stress permissible is about two-thirds of a ton per square inch; the strut has round ends, and $k = 12{\cdot}6$, as in the previous example, and $c = \frac{1}{6250}$

Then, after substituting in equation (73), we find that $A = 107$ square inches, and D, the diameter of jib at centre, $= 12$ in. approximately. Care must be exercised in selecting a piece of timber so that, after being exposed to the weather and wear and tear for a few years, it will not appreciably warp. To provide against such contingencies, the diameter may be slightly increased.

As it is unnecessary to make the diameter at the ends so large as that at the centre, wooden struts are generally tapered both ways from the centre, the most efficient form being a curve of sines. This will be dealt with in detail in the next example.

EXAMPLE 5.—The maximum load that can come upon one of the uprights of a shear legs is 90 tons, and its length is 90 ft. It is made of mild steel plates, butting end to end, riveted together by means of cover strips. To find the diameter of the upright at its middle-section:

The ends are round, and hence $c = \frac{1}{7800}$. The thickness will be small compared with the diameter, and therefore the sectional area will be $2 \pi r t$ square inches, without a sensible error, when r is the radius of the shell and t the thickness. But if ρ is the radius of gyration of the section,

$$A = k \rho^2 = 2 \pi r t,$$

or,

$$k = \frac{2 \pi r t}{\rho^2}.$$

To get ρ in terms of r we proceed as follows: The moment of inertia I_1 of the section about M N (fig. 53) as axis + the moment of inertia I_2 about P Q as axis = the moment of inertia I_3 of the section about an axis passing through R perpendicular to the plane of section.

As $I_1 = I_2$ from the symmetry of the figure, therefore

$$I_1 = \frac{I_3}{2}.$$

Now, $\quad I_3 = \text{area of section} \times r^2$, approximately

$$= 2 \pi t r^3$$

therefore, $\quad I_1 = \pi t r^3 = A \rho^2 = 2 \pi r t \rho^2$

consequently,

$$\rho^2 = \frac{r^2}{2}.$$

Putting this in the expression for k, we have—

$$k = \frac{4 \pi t}{r}.$$

The maximum working stress may be taken as 8 tons per square inch, in compression.

Substituting these values in equation (73), we have—

$$t = \frac{1 + \sqrt{1 + \frac{667}{\;}}}{1{\cdot}12\, r}$$

By giving values to r, we get corresponding values for t, a few of which are appended.

When $r = 15$ in., $t = {\cdot}46$ in.
" $r = 20$ in., $t = {\cdot}31$ in.
" $r = 22$ in., $t = {\cdot}25$ in.

If we select the maximum outside diameter as 30 in., then the requisite thickness is $\frac{1}{2}$ in. The plates butt together, and are secured by covered strips at the joints. In this way the minimum of stress comes upon the rivets.

The maximum tensile stress upon the opposite side to that of maximum compressive stress, equals the stress due to bending only – the compressive stress due to the direct load.

Now $$f_c = \frac{W}{A} = \frac{90}{2\pi r t} = 1{\cdot}9 \text{ tons},$$

and as $$f_c + f_b = f = 8 \text{ tons per square inch},$$

$$f_b \text{ must be } 8 - 1{\cdot}9 = 6{\cdot}1 \text{ tons per square inch},$$

and maximum tensile stress $= f_b - f_c = 4{\cdot}2$ tons per square inch.

This is sufficiently low to allow for rivet holes.

The absence of any bending action at the extremities leaves the material there under the direct crushing stress f_c only. Let d = the diameter of the leg at either end; then

$$W = \pi d t f_c, \text{ or } d = 7 \text{ in.}$$

There are practical difficulties in the way of making the diameter of the ends so small, to say nothing of the æsthetic side of the question; and hence the ends are not made a great deal smaller than the centre, but the plates may then be gradually reduced in thickness from the middle towards the ends. The exact form of the outline is not of great importance. In fig. 53, H F is the centre line of the leg; H G and E F are the radii at the ends, while D C is that at the centre.

If there were no such thing as bending in the leg, then G E would be the contour; but as there is a bending action in addition to the direct crushing action, G C E is the contour; or, in other words, the piece G C E G is put on to resist bending. But the bending moment diagram is a curve of sines [see equation (77)], and, consequently, the moment of resistance diagram also; and if the stress is to be constant throughout the leg (the most efficient distribution of material), the extra sectional area due to bending must also be a curve of sines. Now, the area of cross-section is approximately proportional to the radius, and hence the addition to the radius beyond the quantity H G must be the ordinate of a curve of sines. This very flat curve cannot be distinguished from a parabola or the arc of a circle, and hence the latter may be used without error. The same applies to the flanges of a strut of other sections, as well as an annulus.

CHAPTER XI.

WORKING STRESS.

IN all of the expressions so far obtained to give the dimensions of a piece of material which will sustain the load allotted to it, we find the symbol f, which represents the stress to which the material is to be subjected. As this is by no means a fixed quantity under all circumstances, and as it is a quantity whose numerical value has to be determined by the designer (partly from known results and partly by the aid of his own judgment), some little space will here be devoted to the discussion of its relative values.

To begin with: Suppose we take a piece of material, say wrought iron, and place it in a testing machine, to which is attached an autographic recording apparatus. Now break the specimen in the usual manner, the time during which the load is gradually applied being, say, from one to five minutes. The record of the experiment will be found on the paper of the autographic apparatus, an average copy of which is given in fig. 54, in which extensions are measured to the right from O Y, and the load is measured upwards from O X. The first portion of the load—elongation curve O A—is a straight line, or so nearly straight that it is impossible to detect an appreciable deviation with the scale of diagram there used. The interpretation of this portion of the curve is, that the extension or amount of stretch is proportional to the load which is stretching it. As the line passes through the origin O, it can be represented by the equation—

$$y = m\,x,$$

where y represents the load, x the extension, and m a constant to be determined from the figure. The tangent of the angle A O X $= m$. Divide both sides of the above equation by A, the area of cross-section, and we have—

$$\text{stress} = \frac{\text{load}}{\text{area}} = \frac{m \times \text{extension}}{\text{area}}$$

and
$$\text{strain} = \frac{\text{extension}}{\text{original length}}$$

therefore
$$\text{stress} = \frac{m}{\text{A}} \times \text{strain} \times \text{length} = \frac{m\,l}{\text{A}} \times \text{strain}.$$

But, from equation (23),

$$\text{stress} = \text{E} \,.\, \text{strain} ;$$

hence
$$\text{E} = \frac{m\,l}{\text{A}} = \frac{l}{\text{A}} \tan \text{A O X}.$$

In this way the value of E is obtained. To determine the value of tan A O X with a fair degree of accuracy, the mechanism of the recording apparatus must be so adjusted as to exaggerate the extensions enormously, so that the angle A O X is not much greater than about 45 deg. The line O A is generally called the elastic portion of the diagram, and the upper end of the straight line O A

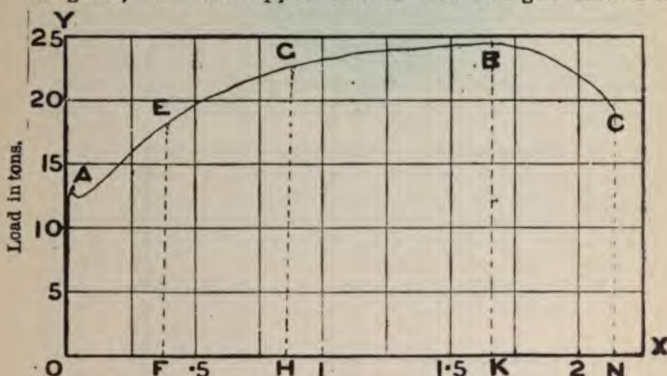


FIG. 54.—Extension in inches.

the limit of elasticity, because at this point the above simple relation of stress to strain ceases. Beyond the point A the extensions increase very much faster in proportion to the load, as indicated by the curve A E G B C. This is generally termed the *semi-plastic* portion of the diagram. It must be evident that after the sudden break in the continuity of the curve at A the material undergoes some change in its constitution. Between A and B the material appears to elongate fairly uniformly along its whole length; but at the stage in the experiment represented by B the specimen begins to give way or elongate, more especially near some one particular cross-section; at the same time contraction takes place very rapidly at the same section. This contraction of area necessitates the specimen

being relieved of some of its load, as shown by the downward tendency of the portion of curve B C. At C the piece fractures.

The load B K is the maximum that can be applied to the specimen, but the *stress* at B is not so great as the *stress* at C, on account of the contracted area of cross-section at C.

If a number of specimens of the same material are broken and their load-extension diagrams compared, it will be found that the elastic portions of all the curves will not differ much from one another, except perhaps in the length of the line O A, but that in all probability the *plastic*

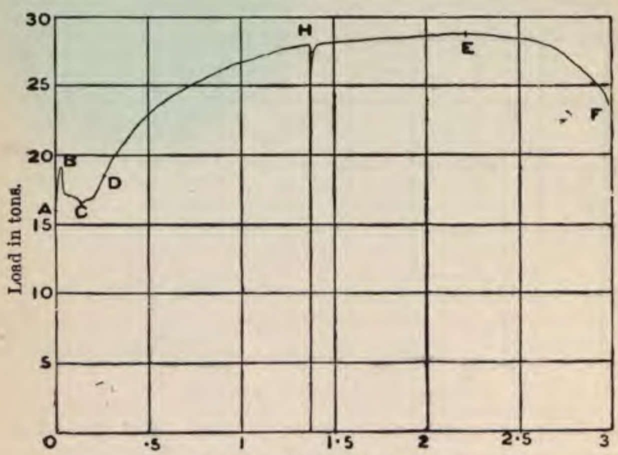


FIG. 55.—Extension in inches.

portions of the curves may show some variation unless the whole of the material is very uniform throughout. The limit of elasticity at A it would be preferable to call the *primitive* elastic limit, in contra-distinction to the higher limits produced by straining, accompanied by permanent elongation, commonly called *set*. If the specimen which was broken, and whose record is given in fig. 54, had been relieved of its load before fracture occurred. the elastic limit would no longer be at A.

For example, assume that the who'e of the load was relieved after the tracing pencil had arrived at E. The pencil would return along the dotted line E F, and after the whole of the load had been removed the specimen would be permanently longer by the quantity O F. The *set* which has been produced is O F. If now the load is re-applied, the pencil will re-trace the line F E, and after arriving at E will proceed along the old curve E G. The same sort of thing will happen if the load is removed after the pencil has arrived at G, O H being the *set* after the removal of the load. On a re-application of the load the straight line H G will be retraced, and then the remainder of the curve, until fracture occurs at C. It will be found on measuring the diagram that E F and G H are approximately parallel to A O, showing that ordinarily the modulus of elasticity E does not vary appreciably, although the specimen is strained beyond the elastic limit. With more sensitive recording apparatus it may be found to vary slightly.

Again, if the load be removed when the tracing point is at E, and then steadily re-applied several times, the tracer will simply move up and down the line E F; or, in other words, although the specimen is permanently longer by the amount of set O F, the range of approximately perfect elasticity has been increased, the limit being now at E instead of at A. The same applies to G H. The material in this new state is *elastically* stronger than in the primitive state.

In dealing with the testing of a specimen it has been assumed that the load has been steadily or gradually applied. Such is generally known as a dead or static load. The total static strength (elastic and semi-plastic) is not affected by the raising of the elastic limit; but the total dynamic strength—*i.e.*, the strength to resist a sudden or live load—is materially affected by a change in the limit of elasticity accompanied by permanent set. Such a load possesses energy, which energy is spent in stretching the specimen. In the primitive state the total work required to be done upon the specimen to break it is represented by the area O A E G C N O, fig. 54. After the piece has received a set O F, the energy required to fracture it is represented by the area F E G C N F, which is less than in the primitive state. Hence the peculiarity that raising the elastic limit by giving permanent set *increases* the static *elastic* strength, but decreases the total dynamic strength of the same specimen.

The stress-strain diagram, fig. 54, is similar to those obtained in commercial testing, where the limit of elasticity and the breaking strength only are required; and a large number of tests have to be conducted in a given time. If the experiment is slightly prolonged, it will be found that the diagram in the region of the limit of elasticity A will undergo a change. The true elastic portion is now O A, fig. 55 From A to B, the strain, though still small, increases more rapidly than the stress, and is accompanied with a small set. At B, a change of constitution takes place (with, probably, a re-arrangement of some or all of the molecules), and we have a larger amount of elongation, without any

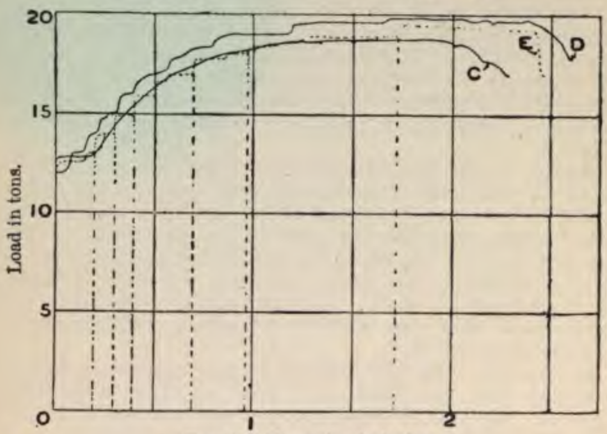


FIG. 56.—Extension in inches.

increase in the load. Between B and D the diagram seems to indicate something like a state of unstable equilibrium, and is generally called the *yield stage;* B being the *yield point.*

Some experimenters (Professor Kennedy among the number) suggest that elongation takes place during the yield stage at different parts of the specimen, more or less successively, and not simultaneously. This stage, too, marks the beginning of semi-plasticity, which continues up to the point E of maximum statical strength; after which contraction takes place locally, and we have a state more nearly approaching to perfect plasticity. If the load is

removed during the semi-plastic state, and re-applied, there occurs a minute set, as shown at H, similar to that from A to B, but the yield stage is not repeated. From this it appears that the yield stage occurs once only during the process of destruction of a bar, but the limit of elasticity will always be followed by a minute set, wherever that limit may be situated in the diagram.

To show the effect of time upon the semi-plastic stage, fig. 56 has been reproduced from that given on page 93 of the eighty-seventh volume of the Proceedings of the Institution of Civil Engineers, and represents some experiments carried out by Professor Unwin. All the

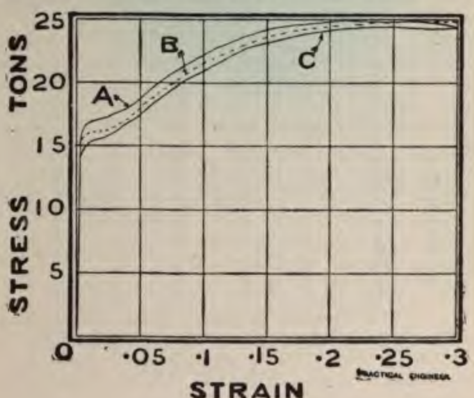


FIG. 57.

specimens were cut from the same bar of Staffordshire iron. The curve C represents the process of destruction of a specimen in the ordinary manner. The curve D represents a similar specimen, but with which a pause of four minutes was allowed after the addition of each ton of load. The curve E represents the record of another similar specimen, when the load was maintained constant for six minutes after the addition of every ton, and then completely removed.

In the same volume above referred to, are details of some similar experiments carried out by Professor Barr on iron wire, in which the rate of application of the load could be varied within almost any limits by allowing sand to flow

into a receptacle which was attached to the wire. The rate of flow of the sand, of course, gives the rate of application of the load. In fig. 57 the curve A is that obtained when the stress was applied at the rate of 5·4 tons per square inch per minute, the curve B at the rate of 1·8 tons per square inch per minute, and the curve C at the rate of ·66 ton per square inch per minute. In fig. 58 the curve A was obtained when the stress was applied at the rate of 1·1 ton per square inch per minute, while in the zigzag curve the load was constant from C to D for 2¼ hours, from E to F for 1 hour, and from G to H for 18 hours.

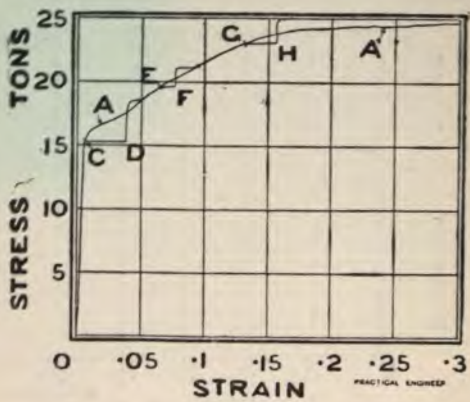


FIG. 58.

The curves in figs. 56, 57, and 58 all show that the duration of the experiment and the rate at which the load is applied have some influence upon the behaviour of a metal beyond its elastic limit, though previous to the elastic limit the influence is inappreciable. They also tend to show that beyond the limit of elasticity the metal behaves in a manner similar to partially plastic material, or a very viscous fluid. By prolonging the experiment, it is generally somewhat easier to perform, and more ductility is generally exhibited. This is the case with material in the semi-plastic state.

Sir Wm. Fairbairn found that if a dead load were imposed on a bar equal to two-thirds of its static breaking load, the bar always fractured after a little time; showing the

effect of time upon the strength of material to be something very perceptible in the long run. This same effect, in a less degree, was shown in figs. 56 and 58.

Probably this time effect forms a part of what is generally termed "fatigue," although imperfect elasticity and viscosity have no doubt a good deal to do with it. Elasticians have not yet been able to formulate a theory upon which they are unanimous, or upon which even a fairly large portion of their number hold similar opinions, as to the nature of the phenomena which generally go by the name of "fatigue of material." Fatigue in general may be briefly described as a decay in strength due to use. A very suggestive and simple experiment for showing the effect of use upon material is that described by Professor Tait in his "Properties of Matter," p. 219, in which a wire is twisted through a right angle to the right, and maintained there for six hours, then to the left through the same angle for half an hour, and finally brought back to its then state of rest. Professor Tait then remarks: "When left to itself it turned slowly towards the right, gradually undoing part of the effect of the more recent twist, then stopped, and twisted still more slowly to the left, thus undoing part of the quasi-permanent effect of the earlier twist. The behaviour of such a wire, strictly speaking, is an excessively complex one, depending, as it were, upon its whole previous history, though, of course, the trace left by each stage of its treatment is less marked as the date of that stage is more remote."

In the Proceedings of the Royal Society of 1865, Lord Kelvin describes an experiment illustrating the cumulative effect of fatigue, in which he maintained wires oscillating for some days. In making a comparison between two wires—one of which had oscillated some days, while the other had only oscillated a few times—he remarks that the arc of the former was reduced to one-half its original value in about 45 vibrations, and that of the latter reduced one-half in 100 vibrations; thus showing the diminution in elastic recovery of the fatigued material. None of the above fatigue was due to straining the wire beyond the elastic limit.

It is easy to imagine the result of such an action in a piece of material subjected to alternating stresses, such as a crank shaft or connecting rod, and it is not unreasonable to expect such a shaft or rod to give way under a load which does not even approach the limit of static strength. Such is by no means an uncommon event.

Some years ago Wöhler set about investigating the behaviour of material under alternating stresses, and although his efforts were not crowned with complete success, he was able to prove experimentally that material so treated could not be expected to resist destruction so easily as material in the primitive state. Speaking roughly, the result of his experiments showed that a load varying from zero to a maximum, and then from the maximum to zero, repeated a very great number of times, may be two-thirds of the static load and still produce the same result. Also that a load equal to one-third of the static load, if allowed to fluctuate between maximum load positive and the same load negative, would produce like results. Approximately, then, the relative values of a static load, a variable load, and an alternating load, are as **1** : 2 : 3.

The chief difficulty in accepting Wöhler's results as truly representative of material in general under the action of a *steadily* fluctuating load is that there exists some doubt as to the actual rate of application of the load. If the load were steadily and slowly applied and relieved, then his results could only be taken to represent the effect of repetition or alternation in altering the constitution of the material; but if instead during the application of the load there occurred any jerks or sudden changes in the rate of application, or if kinetic energy were allowed to be generated by the application of the load, then the rupture of material would in part be due to a kinetic load, and it would be difficult to assign to each component its share in causing destruction.

Of the different formulæ devised for the purpose of representing analytically Wöhler's results, that of Launhardt,* amongst continental writers, and that of Unwin, in England, appear to coincide most nearly with experiment. The latter includes every possible case of stress fluctuation, and until more definite experiments are carried out may be used to determine the breaking stress.

Let f = the static breaking stress—*i.e.*, the stress represented by B K, fig. 54;

f_{max} = the maximum value of the fluctuating stress;

f_{min} = the minimum value of fluctuating stress;

r = the range of fluctuation $= f_{max} - f_{min}$.

* For a discussion of Launhardt's formula, consult "Theory of Structures and Strength of Materials," by Professor Bovey; published by John Wiley and Sons. It is given on page 231 of this work.

Then Professor Unwin gives:

$$f_{\max} = \frac{r}{2} + \sqrt{f^2 - c\,r\,f} \quad . \quad . \quad . \quad . \quad (85)$$

where c is a constant whose value for wrought iron is about 1·42, and 1·66 for steel. The value 1·5 is generally used as the average. For a static load r is zero, and

$$f_{\max} = f.$$

For a load fluctuating between zero and $f_{\max}$, we find

$$r = f_{\max} \text{ and } f_{\max} = \tfrac{2}{3} f$$

approximately; also when the load fluctuates between equal limits on either side of zero,

$$r = 2 f_{\max} \text{ and } f_{\max} = \tfrac{1}{3} f$$

approximately. These results are sensibly the same as those of Wöhler, derived from his experiments. The stress $f_{\max}$ is the *breaking* stress with a fluctuating load, because f has been taken as the *breaking* stress with a steady load. If the *safe* static stress is inserted for f, then $f_{\max}$ represents the corresponding *safe* fluctuating stress.

CHAPTER XII.

Effect of Kinetic Load.

We will next proceed to consider the effect of a "dynamic"—or, more properly, "kinetic"—load. Suppose a *constant* force to act upon a piece of elastic material, such as a rod of iron or a spiral spring, and let O D, fig. 59, represent the accompanying maximum elongation. Also let O C represent the magnitude of this constant force; then the work done upon the material by the force in stretching it is represented by the area of the rectangle O C E D. In the same manner, if D F represent the maximum tension in the material at the instant of maximum elongation, O F D also represents the work done in stretching the material (within the elastic limit). Hence the area O C E D equals the area O F D, and O C = D E = E F, or D F = 2·O C, or, in other words, a kinetic load has at least double the value of a static load of the same intensity.

Proceeding further, assume that at the instant of maximum elongation—*i.e.*, when the end of the rod has

arrived at D—the constant force be reversed in direction; the push on the material when next coming to rest will be represented by 4 O C—thus: The area $O_1 F_1 D_1$ is area O F D repeated. This represents the work done in stretching the material, and hence, when relieved of the stretching force, will be given out by the material in the return stroke. In addition, there is the work done by the force $L D_1$ (= O C) during the whole of the reverse stroke. As the material is elastic, the resilient energy + the work done by force will be again stored in the material at the next coming to rest, which will occur at G_1. The total energy spent during the reverse stroke $D_1 G_1$ = area Z + area of rectangle $L G_1$ = Z

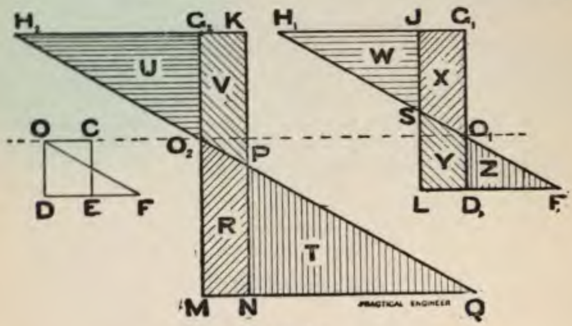


FIG. 59.

+ Y + X. Work done in compressing the material = area of the triangle $H_1 G_1 O_1$ = X + W. Hence

$$X + W = Z + Y + X$$

or

$$W = Y + Z.$$

Therefore, $H_1 G_1 = H_1 J + J G_1 = L F_1 + J G_1 = D_1 F_1 + 2 L D_1 = 2 L D_1 + D F. = 4 O C.$

If the force is reversed again, the pull upon the material at the end of the third stroke equals 6 O C—thus: The triangle $H_1 G_1 O_1$ is repeated at $H_2 G_2 O_2$, the point of no resilience being O_2. The work done by the force during the third stroke is represented by the area V + R. The work done in stretching the material until it next comes to rest is given by the triangle $O_2 M Q$. Therefore we have—

Work stored in material at beginning of stroke + work done on material during the stroke = work done in stretching material,

or $$U + V + R = R + T,$$

and $$U + V = T;$$

that is, $M Q = M N + N Q = M N + H_2 K = 2 M N + H_2 G_2$
$= 2.O C + H_1 G_1 + 2.O C + 4 O C = 6.O C.$

Similarly at the end of the nth repetition of the load, the corresponding tension or thrust in the material is—

$2n$ times the load or straining force.

It is evident that after a very short time the stress in the material will become so great that rupture will occur. For

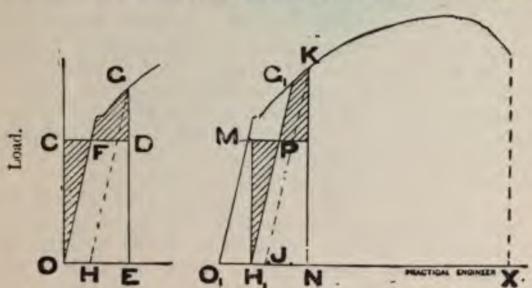


Fig. 60.—Extension.

the above state of stress to exist, the time of completion of a stroke must coincide with the period of vibration of the material itself. This does not often occur, though it is quite possible. This same problem has been solved in a different manner by Mr. C. H. Innes, in his treatise on "Problems in Machine Design," page 102.

If the alternating live road is applied, not at the *beginning* of each stroke, but after the material has returned to a state of rest at its position of no stress, the resulting maximum stress produced is evidently double that produced by a static load of the same intensity in either direction.

Material subjected to a kinetic load, of intensity equal to half the static breaking load, must fracture after some number of repetitions of the load. This is easily demonstrated; for, referring to fig. 60, the curve O F G is the

load elongation diagram (the inclination of O F being exaggerated). The intensity of the kinetic load is represented by O C, and the corresponding elongation by O E. As the energy of the kinetic load equals the work done in stretching the material through O E, the area O C D E = the area O F G E, and consequently the area O C F = the area F G D. After the removal of the load the material has a permanent set O H. The load being again repeated, we have the area H_1 M L N = the area H_1 G_1 K N, and consequently the area H_1 M P = the area P G_1 K L, the amount of elongation being H_1 N measured from the position of first rest. The amount of elongation beyond the previous elongation O E is the horizontal distance between G_1 and K, and the total set after the second repetition is O H + H_1 J = O_1 J. Each repetition of the load will be accompanied by additional permanent set; and hence when the set has accumulated to an amount equal to the elongation O_1 X, corresponding to the rupture point, the material is on the verge of giving way if other circumstances have not previously produced rupture. Here the load is assumed to be always of the same kind.

It may therefore be concluded that a repeated kinetic load should never be so great as half the static breaking load, if the structure is to be maintained in tact.

To take another case, let the material be loaded with a static load F_1, represented by P Q, fig. 61, O Q being the corresponding elongation. The area O P Q represents the work done during elongation. Now, suppose that the material in the strained condition is subjected to an additional kinetic load F_2 of the same kind as F_1; the elongation is now increased from O Q to O R, and the maximum pull or thrust in the material is represented by J R. Total work done by stretching forces = area O P Q + area P S R Q + area P S V T. The total work done in stretching the material = area O J R. Therefore

$$\text{area } OJR = \text{area } OPQ + \text{area } TVRQ.$$

And after taking away the parts common to both sides of the equation, we have left the

$$\text{area } JMV = \text{the area } TMP,$$

and consequently

$$JV = TP = VS = F_2;$$

therefore
$$JR = 2F_2 + F_1.$$

Now, elongations are measured vertically downwards from O, and tensions horizontally to the right from O R; therefore J R represents the maximum tension (or thrust, as the case may be) produced in the material by the primitive load F_1, and the afterwards applied kinetic load F_2 conjointly. The effective load is $F_2 + F_1$ if both act in the same direction; but if in opposite directions, it is $F_2 - F_1$. Then the ratio—

$$\frac{\text{maximum tension}}{\text{effective load}} = \frac{2F_2 \pm F_1}{F_2 \pm F_1} = \frac{RJ}{RV}$$

If the total load is purely a dead load, F_2 is zero and the ratio is unity; but should the total load be wholly kinetic, F_1 is zero and the resulting ratio is 2, a result previously obtained in connection with fig. 59. Again, if the kinetic

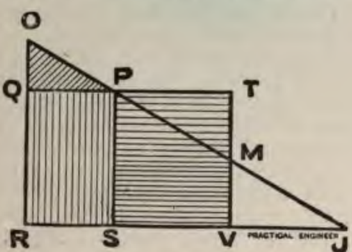


FIG. 61.

and static loads are of the same intensity, and in the same direction, then the ratio is $\frac{3}{2}$, and lastly, should the two loads act in opposite directions, and their algebraical sum be numerically equal to the primitive load F_1, then the

$$\text{effective load} = F_2 - F_1 = F_1,$$

and hence $$F_2 = 2F_1;$$

also the ratio $$\frac{2F_2 - F_1}{F_2 - F_1} = 3.$$

But although this effective load $F_2 - F_1$ produces the maximum result in the way of destruction, it is seldom in actual practice that we find loads imposed upon structures in the same manner. The most common way of applying an alternating live load is as a simple load alternately in either direction, from the position of zero strain, similar to the

pressure on the piston of a double-acting engine. In this way the maximum effect of a simple alternating live load is twice that of a dead load.

The above results have been obtained on the assumption that the material has not been strained beyond its elastic limit, but the same applies also beyond the elastic limit, if the load is repeated a great number of times. This is obvious, for each time the elastic limit is exceeded it is also raised in value, and the process may go on until rupture occurs, though, of course, the material is at the same time being permanently deformed to an enormous extent, far greater than could generally be allowed in practical construction.

The great increase in extension after the elastic limit is passed virtually precludes any material being strained beyond that limit, during the *working* life of the material; though at the elastic limit the load is only from one-half to two-thirds of the maximum possible load.

In his admirable treatise on "Bridge Construction," Professor Fidler deprecates the use of the term "fatigue," as applied to the deterioration of the strength of material; at the same time he enthusiastically advocates the application of the "dynamic" theory in place of that of fatigue. The actual numbers* representing that fraction of static stress at which material subject to fluctuation stresses should fracture, as deduced from Wohler's experiments, are strikingly similar to those deduced from the dynamic theory, but at the same time the majority of authorities on strength of materials do not by any means appear to be convinced that fatigue does not exist.

An attempt has here been made to point out the direction in which to look for guidance in determining the safe working stress to which any particular piece of material may be subjected. In general, the number of brands of

	* Breaking stress as deduced from—	
	Wohler's experiments.	Dynamic theory.
Steady loads of say W tons	f.	f.
Load fluctuating between zero and W	$\frac{2}{3}f$.	$\frac{1}{2}f$.
Load fluctuating between W and − W	$\frac{1}{3}f$	$\frac{1}{3}f$

iron, and the liability of steel to variation by very slight variations in its composition, prevent any definite number being given as the absolute strength of any particular metal; but there is here appended a table of average values of strength, derived from the most recent practice. The values of the working stress are those when the material is well protected from deteriorating influences, such as rust, corrosion, pitting, &c., and should there be a chance of any of

Material.	Nominal Breaking stress in tons per square inch.			Stress at yield point in tons per square inch.			Working stress for a dead load in tons per square inch			Weight of one cubit foot of material in lbs.
	Tension.	Compression.	Shear.	Tension.	Compression.	Shear.	Tension.	Compression.	Shear.	
Wrought iron..	22	22	18	14	14	10	7	7	5	480
Mild steel ...	28	28	24	18	18	..	9	9	7·5	480
Cast iron	8	40	10	4·5	*	3·5	2	12	1·5	450
Oak	5	3	2	..	..	..	..	..	..	50
Pine	1·5	1·5	1	..	..	..	..	..	..	44

NOTE.—The tensile resistance of wrought iron across the fibres is approximately ·85 of that with the fibres; the resistance given in the column headed "Compression" is that of short specimens only. The working stress in torsion may be taken as three-fifths of the shearing stress in the above table. The working stress for a load fluctuating between zero and a maximum, repeatedly applied, should be about two-thirds of the dead load stress; and the stress for a load fluctuating between equal positive and negative values should be one-third of the dead load stress. Annealed cast steel has a strength about 10 per cent greater than mild steel. The shearing stress of oak and pine was measured with the grain of the material.

The bearing pressure for rivet and pin joints may be taken as not greater than twice the shear stress in the material; the area over which the bearing stress is reckoned being that of the diametral section of the pin or rivet. Parts of a structure that cannot well be covered with some preservative, such as paint, and those parts subject to deterioration, should be designed with a less working stress than that given in the above table.

these occurring, a corresponding decrease in the working stress must be made.

There are other points in connection with the working stress which only experience can determine, and which are out of the province of such notes as these. Readers who are

* There is no well-defined yield point for cast iron, but 30 tons per square *inch can be* used for strut failure.

especially interested in this part of our subject should refer to the larger treatises (some of which have been mentioned), but more especially to the Proceedings of the different engineering institutions.

With reference to the table here given, it must be clearly understood that the values contained therein are only *average* values; and that one of the larger works on the strength of materials must be referred to for obtaining information regarding the idiosyncrasies of material under the application of stress. For the web bracing of bridges, the most economical distribution of material will be obtained by finding the limits of fluctuation of stress, and designing accordingly.† For a full discussion of this part of our subject, the reader is referred to "A Practical Treatise on Bridge Construction," by T. Claxton Fidler, M.I.C.E.

Sir Benjamin Baker, in commenting upon the action of a rolling load upon the cross girders of railway bridges,* intimates that although the load is apparently imposed suddenly, experience seems to indicate that it is not of the class generally termed as a live load, and that the excess of effect above that of the ordinary dead load may be safely met in ordinary cases by adding 20 per cent to the dead-load value of the load, and decreasing the working stress by 20 per cent. This is roughly equivalent to taking the value of the (rolling) load in such cases as about one and a half times its dead-load value. This is only one of many cases where it is necessary to carefully examine into the real nature of the load as far as possible.

No reference has been made to so-called factors of safety. The term is to a great extent a misnomer. The margin of stress, in the above table, which is set aside to counterbalance faulty workmanship, errors of judgment, unobservable or other sources of weakness, is in amount about equal to the working stress itself; thus, the limit of elasticity of wrought iron is about 14 tons, and the working stress about 7 tons per square inch. But although in a great many cases material put into a structure in the primitive state would be of little use after being strained beyond the elastic limit, on account of the great distortion of the structure, and consequent virtual failure for *working* purposes, yet the same structure thus strained does not approach the limit of *strength*, and will not utterly collapse, with possible injury to life or limb. While the *working* factor will generally be the elastic limit of stress divided

* "Short-span Railway Bridges." By B. Baker. E. and F. N. Spon.
† See page 231.

by the working stress of the dead load or the dead-load equivalent, the factor of *safety* will be the limit of strength divided by the working stress of dead load or its equivalent.

CHAPTER XIII.

ROOF TRUSSES.

THE load upon a roof truss is due to the covering, the principals, purlins, and rafters, and snow and wind. Taking the covering first, the following are average values which are found in actual practice. The numbers represent the weight of covering per square foot of area to be covered in.

Slate	8 lb.
Wood, 1 in. thick	3·5 lb.
Lead	8 lb.
Corrugated iron, 16 B.W.G.	3·5 lb.
Stone tiles	24
Plain pantiles	18

Slates should not be used upon a roof if the slope is less than 1 in 3, as the rain then soaks between the slates.

The average weight of common rafters is 3 lb. per square foot of surface, while the weight of a timber frame for an ordinary roof is about 6 lb. per square foot.

Messrs. Johnson, Bryan, and Turneaure give the weight of ironwork about a roof with iron principals and purlins as—

$$\left(\frac{\text{span in feet}}{25} + 4\right) \text{ lb. per square foot of surface covered,}$$

and the weight of a complete iron truss or principal as $\frac{1}{24} b\, l^2$, where l is the length of span in feet and b the distance between trusses, the weight being in pounds.

The width apart of trusses may be about one-fifth of the span, but the larger it is in general the more economical.

The weight of snow varies greatly, according to situation and locality. In England it does not exceed 6 lb. per square foot, in mid-Europe 15 lb. per square foot, and in America 30 lb. per square foot. In England the snow and wind will not together strain a roof truss, because the wind blows the snow off, if there is any quantity. Hence it is often the

* See Articulated Structures in the Appendix.

practice to assume a load of 30 lb. per square foot over the whole surface as representing the maximum effect of the wind and snow.

* As regards the pressure of wind alone on a surface, its maximum value is about

$$\frac{v^2}{400} \text{ lb. per square foot,}$$

where v is the velocity in feet per second perpendicular to the surface. In general, the direction of the wind is approximately horizontal, and the expression most often used to give the pressure per square foot normal to any surface is that due to Hutton, namely—

$$P_n = P (\sin \theta)^{1 \cdot 84 \cos \theta - 1}$$

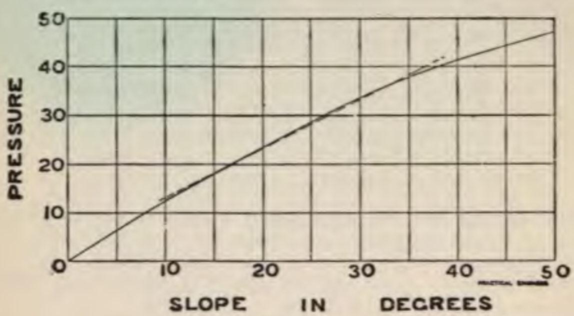


Fig. 62.

Where P is the pressure per square foot on a surface, perpendicular to the direction of the wind, P_n is the normal pressure per square foot on the surface, and θ is the inclination of the surface to the direction of wind—*i.e.*, to the horizon. The maximum pressure per square foot in England is about 56 lb. per square foot on the surface, perpendicular to the direction of the wind. The above equation has been plotted in fig 62 with ordinates representing normal pressures per square foot in pounds, and abscissæ representing the inclination of the surface of roof to the horizon in degrees.

Although in England the Board of Trade requires bridges to be designed with an assumed wind pressure of 56 lb. per square foot, it is only under very exceptional conditions that this pressure is ever realised, even on very small portions of the surface exposed to the action of the wind.

* See Appendix.

Not only does the form of the structure materially affect the magnitude of the wind pressure, but the form and disposition of structures in the immediate neighbourhood exert an enormous influence in modifying and sometimes annihilating it altogether. Some experiments recently carried out by Professor Kernot in Australia show this in a marked degree. A strong blast was created by a mechanically-driven fan, and the pressures on different objects compared with those on a thin plate normal to the direction of the current of air. The ratio of the pressure on the object to that on a thin normal plate he called the modulus of the object. The modulus for rectangular blocks was the same whether the surface was normal to the direction of current or not, and was equal to about ·8 on the average. Cylinders gave a modulus of about ·5 when they were alone in the current.

The results of experiments with model roofs showed that if there were vertical walls supporting the roof, the pressure due to the wind was considerably reduced. In the case of a roof whose slope was 45 deg., the presence of the walls reduced the pressure on roof by 80 per cent, and with a 30 deg. slope the pressure was inappreciable. The pressure on lattice work was found to approach that on the complete surface if the interstices were small, while one girder of a bridge tended to shield its companion on the leeward side, though the amount of shielding depended upon their distance apart.

Athough 30 lb. per square foot of surface is probably ample to allow for both wind and snow, Hutton's rule will be used in the first two or three examples of roof trusses that are worked out, using 56 lb. per square foot as the maximum possible wind pressure. From what has been said above it is evident that the design of the crescent-shaped roof that follows the French truss will be in general much too heavy. This also applies to the king rod truss.

French Truss. *

A span of 70 ft. (centre to centre of bearings) has to be covered by a French truss roof, and the centre tie rod must be at least 4 ft. 2 in. above the horizontal line joining the centres of the extreme joints. The truss to be made of wrought iron, and pin joints to be used throughout, if possible without inconvenience. It is required to design the roof, which has to be covered with slate and wood sheathing, and is amply protected from wind. It rests upon walls.

* See Articulated Structures in the Appendix.

The truss is shown in skeleton form in the upper part of fig. 63, and one side of the same with wood and slate covering, and purlins in the lower part of the same figure. The fixing of the centre lines from the given data may be

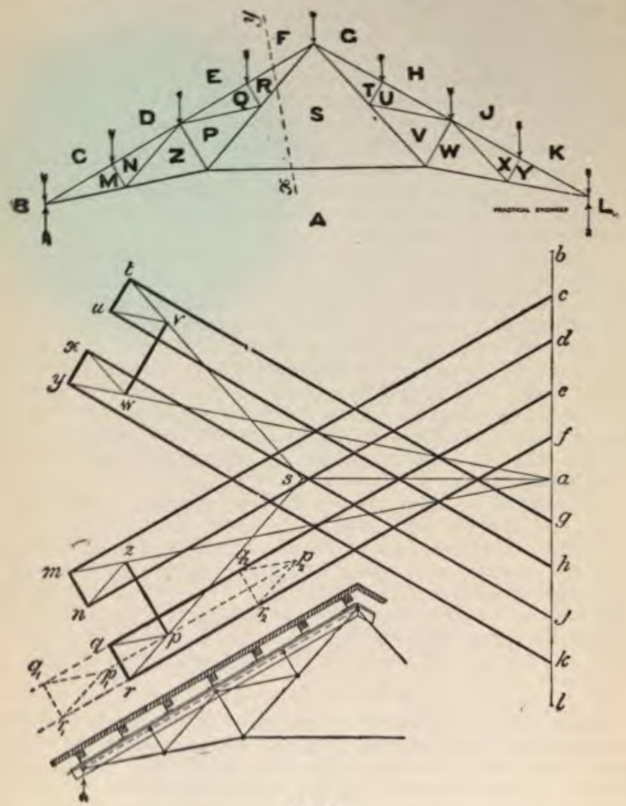


FIG. 63.

accomplished as follows: The lines of the two supporting forces A B and A L are set down to some convenient scale, 70 ft. apart, with their points of application on the same

level. The angle of slope, 30 deg., is then set down at both ends, the intersection of the two bounding lines being at the point of application of the force F G—that is, the vertex of the roof. The length of each slope is then bisected by the lines Z P and V W respectively, and the intersections of these lines with the horizontal A S, at a height of 4 ft. 2 in. above the level of the points of application of the supporting forces, give the vertices of the side trusses. The remainder of the lines follow at once. The weight of the roof per square foot of area covered by the roof is on the average—

Slate		8 lb.
Wood lining, 1 in. thick		4 lb.
Snow		6 lb.
Iron $= \frac{\text{span}}{25} + 4$	=	7 lb.
Total		25 lb.

The principals may be spaced 17 ft. apart, and consequently the total load upon one principal is $70 \times 17 \times 25$ lb. = 13 tons approximately, and the total load upon one side truss is $6\frac{1}{2}$ tons. This is divided equally between seven purlins, as shown in fig. 61, the purlins being placed so as to divide each panel into two parts in the ratio of 2 to 1. In this way the extreme end joint will sustain (approximately) one-seventh of $6\frac{1}{2}$ tons plus one-third of the weight upon the second purlin—that is, $(\frac{1}{7} + \frac{1}{3} \text{ of } \frac{1}{7})$ of $6\frac{1}{2}$ tons, which equals 1·24 ton. The second joints from the top and bottom also sustain a like load. The middle joint sustains the load on one purlin, together with one-third of the load of the purlins on either side of it, or in all 1·5 ton. And, finally, the top joint will sustain double the amount of either of the extreme end joints—that is, 2·5 tons. The stress diagram, fig. 63, may now be drawn, and the several stresses scaled off and tabulated as in the table given on page 142.

* In drawing the stress diagram some little difficulty may be experienced by the novice in finding the point *p*. The stresses *c m*, *a m*, *m n*, *n d*, *n z*, and *a z* follow immediately from the previous work on stress diagrams, but it is at first sight impossible to locate the point *p* by following out the usual methods. The difficulty may be overcome in a number of ways, a couple of which may be given here, as they will be found advantageous in deducing the stresses in other indeterminate forms. The direction of *z p* is known, and

* See Appendix.

the problem will be completely solved, if we can find geometrically the point p, or if the magnitude of $z\,p$ be found.

From the symmetry of the figure, p will occupy a similar position in the triangle p, q, r to that of the point z in the triangle $m\,z\,n$. The point q is not yet found; hence, assume any point q_1, in $e\,q$ produced, and through q_1 draw $q_1\,r_1$,

Table of Quantities, French Truss.

Member.	Total stress in tons.	Composition of members.		
		Length in inches.	Sectional area in sq. inches.	Make up.
AB	6·5	—	—	—
AM	+ 14·3	130·00	2·0	Two round bars $1\frac{1}{8}$ in. diameter.
CM	− 16·3	120·75	—	—
MN	− 1·1	41·25	—	—
DN	− 15·6	120·75	—	—
NZ	+ 1·7	130 00	·25	One round bar $\frac{5}{8}$ in. diameter.
AZ	+ 12 6	130·00	1·8	Two round bars $1\frac{1}{16}$ in. diameter.
ZP	− 4·5	82·50	—	—
PQ	+ 1·7	130·00	·25	One round bar $\frac{5}{8}$ in. diameter.
EQ	− 14 8	120·75	—	—
QR	− 1·1	41·25	—	—
SP	+ 6·0	130·00	·85	One round bar $1\frac{1}{16}$ in. diameter.
SR	+ 8·7	130·00	1·1	Two round bars $\frac{7}{8}$ in. diameter.
RF	− 14·2	120·75	—	—
AS	+ 7·2	340·00	1·1	One round bar $1\frac{1}{4}$ in. diameter.

intersecting $f\,r$ in r_1, this being parallel to Q R. Then through q_1 and r_1 draw $q_1\,p_1$, and $r_1\,p_1$ parallel to Q P and R P respectively, intersecting in p_1. Next select any other point q_2, and draw the triangle p_2, q_2, r_2. The line $p_1\,p_2$ will contain the vertices of all the triangles, p, q, r for every possible value of $e\,q$; and hence the intersection of $p_1\,p_2$ with $z\,p$ gives the required point p, and the stress diagram may now be completed.

We may, if we like, resolve all the forces acting at the point E D N Z P Q in two directions, along and perpendicular respectively to E Q. The components parallel to E Q cannot affect the stress in P Z, because they are at right angles to it; therefore consider only the perpendicular components. By the first law of equilibrium the sum of these components must be zero. The stress in N Z is tensile, and from symmetry we should expect to find that in P Q tensile also. That it is tensile may be easily shown thus: Because P S and R S are in the same straight line, there can be no resultant force acting at the hinge joining them in a direction perpendicular to either of them, and consequently the stress in P Q must be of the opposite kind to that in

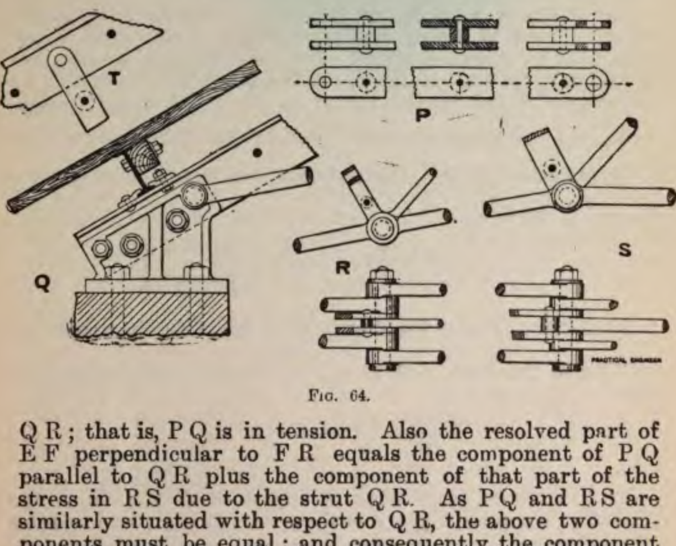


FIG. 64.

Q R; that is, P Q is in tension. Also the resolved part of E F perpendicular to F R equals the component of P Q parallel to Q R plus the component of that part of the stress in R S due to the strut Q R. As P Q and R S are similarly situated with respect to Q R, the above two components must be equal; and consequently the component of the stress in P Q parallel to Z P is one-half of E F. The stress in N Z is known, and therefore its component is known, and equals one-half the component of C D parallel to M N. Now, as the three components of $d\,e$, $n\,z$, and $p\,q$ are all tending to shorten Z P, then Z P must be a strut, and the total stress in it must be equal to the sum of the three components. Hence the point p is found.

Another way of getting over the difficulty is by the "Method of Sections." Cut the truss by the line xy (shown dotted), and replace the right-hand portion of the structure by forces which will balance the stresses in the three members which are cut. The portion of structure which remains is maintained in equilibrium by the forces $a\,b$, $b\,c$, $c\,d$, $d\,e$, $e\!f$, $f\,r$, $r\,s$, and $s\,a$; and the sum of the moments of these forces about any point must be zero. Take the vertex of the roof as the point. The moments of the forces $f\,r$ and $r\,s$ vanish, and the only unknown force is $a\,s$. Having obtained this force, the diagram may now be completed. In the figure the compressive stresses have been denoted by thickened lines. This is merely a matter of choice, and does not affect the drawing of the diagram.

The struts M N, Q R, and Z P may be conveniently made up of a pair of strips riveted together, the strips being separated by distance pieces. An elevation and section are given at P, fig. 64. If flexure takes place, it must do so in the plane of the truss. Let d be the width of each strip, and $\frac{b}{2}$ the thickness of same. Then substituting in equation (73), remembering that the ends are round, and consequently $c = \frac{1}{8600}$, we have—

$$A = \frac{A_o}{2}\left[1 + \sqrt{1 + \frac{4\,k\,l^2}{8600\,A_o}}\right]$$

Then using 7 tons for f and $\frac{12{\cdot}2\,b}{d}$ for k*, at the same time putting in the load on Z P and its length in inches for l, we have—

$$\left(\frac{b\,d}{\cdot 32} - 1\right) = 1 + 59{\cdot}5\frac{b}{d}$$

If $\frac{1}{2}$ in. strips were used, b would be 1, and from the equation we should get $d = 2$ approximately; and making

* $b\,d = A = k\,\rho^2$ and $d = n\,\rho = 3{\cdot}5\,\rho$

therefore $b\,d = k\,\frac{d^2}{3{\cdot}5^2}$

or $k = \frac{12{\cdot}2\,b}{d}$.

a small allowance for the rivets, we may put $2\frac{1}{4}$ in. as the minimum width of the strips.

In the same way the minimum width of the smaller struts M N and Q R is found to be about 1 in., if the thickness is assumed to be $\frac{3}{8}$ in. Then, if we allow for the rivets, the width may be taken as $1\frac{1}{4}$ in.

Next, to find the thickness of the distance pieces. Let x be the distance between the strips—that is, the thickness of the distance piece at the middle of strut. Then the radius of gyration (of the cross-section) perpendicular to the plane of the truss must not be less than that in the plane of the truss.

In the plane of truss—

$$\rho = \frac{d}{3{\cdot}5} = \frac{2\frac{1}{4}}{3{\cdot}5} = {\cdot}64.$$

Perpendicular to the plane of truss—

$${\cdot}64 = \rho = \frac{\mathrm{D}}{n}$$

and in this form of section, perpendicular to the plane of truss, $n = 3$. Hence $\mathrm{D} = 2$ approximately, where D is the depth of strut perpendicular to the plane of the truss. As the thickness of each strip is $\frac{1}{2}$ in., the thickness of the distance piece must be not less than 1 in. In the same way the distance piece for the smaller struts should not be less than $\frac{1}{2}$ in. thick.

The upper member of each side truss is divided into four equal parts, the lower one of which (C M) is the most heavily stressed. Now, C M is, when stressed, approximately perpendicular at its upper end to M N, and hinged at its other end; consequently, it is most nearly similar to a strut with one end fixed and the other round. The strength of this kind of strut is midway between that of a strut with both ends fixed and that of a strut with both ends round; therefore, $c = \frac{1}{17200}$. In a tee iron, with the leg longer than the cross-piece, the value of k is about 2·5, and n is roughly about 3. Inserting these values in equation (73) or (61), the area A of cross-section is about $4\frac{3}{4}$ square inches. The largest tee iron in Messrs. Dorman and Long's catalogue is 6 in. × 3 in. × $\frac{1}{2}$ in., having an area of cross-section of about $4\frac{1}{4}$ square inches.

This may be used by reducing slightly the load on each principal—that is, by reducing the distance between principals. A reduction of a couple of feet would be ample.

On the other hand, a tee iron can easily be made up of a pair of angle irons riveted together. In this way the two angles may be each $4\frac{1}{2}$ in. × $2\frac{1}{2}$ in. × $\frac{3}{8}$ in., giving together a section of 5 square inches. This arrangement will be here adopted. The value of the bending action of the purlins situated between the joints is exceedingly small, and the equivalent stress in this case would not amount to more than 3 per cent of the total stress, and so can be neglected in calculating the strength of the struts.

Lastly, the size of the purlins have to be decided upon. These will be bolted to the principals, and will (though not necessarily) be of such a length as to cover two bays of the roof. Seven purlins on each side have been suggested, and hence the load sustained by each purlin per bay will be just about 1 ton, and it will be evenly distributed along the purlin. Let W be this weight; then the maximum bending moment at the centre will be $\frac{Wl}{8}$, where l is the distance between principals.* This bending moment equals the moment of resistance of the purlin, which equals $\frac{fI}{h}$. [See equation (29)].

Now, makers of angle irons and joists give, in their trade catalogues, the weight per foot, the size of cross-section, and the safe load which can be uniformly distributed over a piece of the same material 1 ft. in length. This last information saves much calculation in selecting any particular section. Thus let w be the load that a joist 1 ft. in length will safely carry uniformly distributed; then the maximum bending moment is $\frac{wl}{8} = \frac{w}{8}$, as l is 1 ft. The bending moment $= \frac{fI}{h}$; hence we have—

$$\text{From the catalogue, } \frac{w}{8} = \frac{fI}{h};$$

and

$$\text{from the purlin, } \frac{Wl}{8} = \frac{fI}{h}.$$

Now, the right-hand sides of both of these equations are the same, for the stress f is the same in both; the moment of inertia must be the same, as the two pieces of material are taken from the same piece, and consequently h must be the

* The purlin is here considered as a beam supported at each end, and uniformly loaded.

same in both. Therefore the left-hand sides must be equal, or

$$w = \mathrm{W}\,l$$

Hence multiply the weight on each purlin per bay by the distance between principals, and look in the trade catalogue for the distributed load $w = \mathrm{W}\,l$ that can be safely carried by a piece 1 ft. long when uniformly distributed. $\mathrm{W}\,l = 1 \times 17$ tons-feet, and the nearest section that will sustain 17 tons over 1 ft. is the $4 \times 1\frac{3}{4}$ in. rolled joist, weighing 8 5 lb. to the foot.

As an aid in future designs, there is appended a table of sizes of irregular-shaped angle irons, and another table giving the above-quoted dimensions and weights of rolled steel joists. These have been kindly furnished by Messrs. Dorman and Long, of Middlesbrough, and will be found in all their trade catalogues.

TABLE OF DIMENSIONS OF ANGLE IRONS (UNEQUAL SIDES).

2 by	$1\frac{1}{2}$ by	$\frac{3}{16}$ to	$\frac{3}{8}$	$4\frac{1}{2}$ by	4 by	$\frac{3}{8}$ to	$\frac{7}{8}$
$2\frac{1}{2}$	2	$\frac{1}{4}$	$\frac{1}{2}$	5	3	$\frac{5}{16}$	$\frac{3}{4}$
3	2	$\frac{1}{4}$	$\frac{9}{16}$	5	$3\frac{1}{2}$	$\frac{5}{16}$	$\frac{3}{4}$
3	$2\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{8}$	5	4	$\frac{3}{8}$	$1\frac{3}{8}$
3	$2\frac{3}{4}$	$\frac{1}{4}$	$\frac{11}{16}$	$5\frac{1}{2}$	3	$\frac{3}{8}$	$\frac{3}{4}$
$3\frac{1}{2}$	$1\frac{3}{4}$	$\frac{5}{16}$ &	$\frac{1}{4}$	$5\frac{1}{2}$	$3\frac{1}{2}$	$\frac{3}{8}$	$1\frac{3}{8}$
$3\frac{1}{2}$	$1\frac{3}{4}$	$\frac{3}{8}$ &	$\frac{5}{16}$	$5\frac{1}{2}$	4	$\frac{3}{8}$	$\frac{7}{8}$
$3\frac{1}{2}$	$1\frac{3}{4}$	$\frac{7}{16}$ &	$\frac{5}{8}$	6	3	$\frac{3}{8}$	$\frac{3}{4}$
$3\frac{1}{2}$	$2\frac{1}{2}$	$\frac{5}{16}$	$\frac{5}{8}$	6	$3\frac{1}{2}$	$\frac{3}{8}$	$\frac{7}{8}$
$3\frac{1}{2}$	3	$\frac{5}{16}$	$\frac{11}{16}$	6	4	$\frac{3}{8}$	$\frac{7}{8}$
4	$2\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{4}$	6	5	$\frac{7}{16}$	1
4	3	$\frac{5}{16}$	$\frac{3}{4}$	$6\frac{1}{2}$	3	$\frac{3}{8}$	$\frac{3}{4}$
4	$3\frac{1}{2}$	$\frac{3}{8}$	$\frac{7}{8}$	$6\frac{1}{2}$	4	$\frac{7}{16}$	$1\frac{3}{8}$
$4\frac{1}{2}$	$2\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{4}$	$6\frac{1}{2}$	$4\frac{1}{2}$	$\frac{7}{16}$	1
$4\frac{1}{2}$	3	$\frac{5}{16}$	$\frac{3}{4}$	7	3	$\frac{3}{8}$	$\frac{3}{4}$
$4\frac{1}{2}$	$3\frac{1}{2}$	$\frac{3}{8}$	$\frac{7}{8}$	7	4	$\frac{7}{16}$	$\frac{7}{8}$

In addition to these tables there is also given a table in which may be found the moment of inertia of the usual sections, together with the square of the radius of gyration (ρ^2), the average value of the coefficient k in the equation $\mathrm{A} = k\,\rho^2$, the average value of the coefficient n in the equation $n = d$, and the quotient moment of inertia divided by the distance of the neutral axis from the extreme fibres, or $\mathrm{I} \div h$. The column giving values of k will be found useful with equation (73). (See pages 152 and 153 for table.)

Some details of the several joints are shown in figs. 64 and 65. The bearing shoe is shown in elevation at Q, together with the purlin in section. At R is shown a plan and elevation of the joint A M N Z, fig. 63; and the joint A Z P S is given in plan and elevation at S, fig. 64. A strut is also

shown at P and T. In fig. 65 is to be found an elevation of

TABLE OF STRENGTHS OF ROLLED STEEL JOISTS.

Weight per foot length in pounds.	Normal section in inches.	Distributed load in tons that 1 ft. will safely carry when the working stress is one-third, one-fourth, and one-fifth of the breaking stress.		
		One-third.	One-fourth.	One-fifth.
5	3 by 1¼	8·2	6·19	4·95
0·25	3 3	20·56	15·42	12·33
6	3½ 1½	12·35	9·26	7·41
10·75	8½ 3	25·46	19·1	15·27
8·5	4 1¾	19·5	14·6	11·7
12·75	4 3	33·13	24·85	19·88
14	4⅝ 3	41·15	30·79	24·63
6·5	4¾ 1¾	19·4	14·5	11·6
9·25	4¾ 1¾	24·2	18·15	14·52
13	5 3	44·07	33·05	26·45
15·25	5 3	48·85	36·64	29·31
17	5 4 13/16	59·44	44·58	35·66
23·75	5 4½	77·72	58·29	46·63
25·5	5 5	86·74	65·05	52·04
9	5¼ 1½	23·51	17·64	14·11
11	5¼ 2	34·69	26 01	20 81
12·25	6 2	40·57	30 43	24·34
13	6 3	50·92	38 19	30·55
16	6 3	63·21	47·4	37·92
19	6 4½	80·35	60·26	48·21
26	6 5	103·93	77·94	62·36
18	6¼ 3½	75·35	56 52	45·21
20	7 3¾	91·2	68·4	54·72
18	8 4	105·03	78·77	63·01
25	8 4	131·86	98·89	79·11
31·25	8 5	171·7	128·78	103·02
36	8 6	202·92	152·19	121·75
24·25	9 3¾	135·11	101·33	81·07
58	9 7	345·32	258·99	207·19
37	9 13/16 4½	227·13	170 35	136·28
31 5	10 4½	212·01	159·01	127·2
29	10 5	210·13	157·6	126·1
35	10 5	258·25	193·68	154·95
45·5	10 6	306·45	229·84	183·87
43	12 5	322·75	242·06	193·66
32·4	12 5	272·28	204·21	163·37
54	12 6	457·96	343·47	274·78
40	13 5	333·36	250 02	200·02
45	14 6	438·47	328·85	263·08
57	14 6	511·51	391·97	313·57
42	15 5	422·8	317·1	254
61	15 6	608·2	456·15	364·9
50	16 5	507·26	380·58	304·46
64·5	16 6	649·87	487·41	389·92
75	18 7	909·2	681·9	545·4

Of the above joists, lengths varying from 10 ft. to 40 ft. are generally kept in stock of sizes from 6 in. by 4½ in. upwards, and the remainder in lengths varying from 10 ft. to 30 ft.

the junction plates for the vertex of a principal. The

small vertical rod is for the purpose of supporting the member A S and preventing it from sagging. In the upper part of fig. 65 is a detail of the joint D E Q P Z N. These details do not, of course, belong exclusively to this form of truss, but may be used where convenient.

The upper cord or principal rafter, fig. 63, when of great length, is generally stayed laterally by cross-bracing to

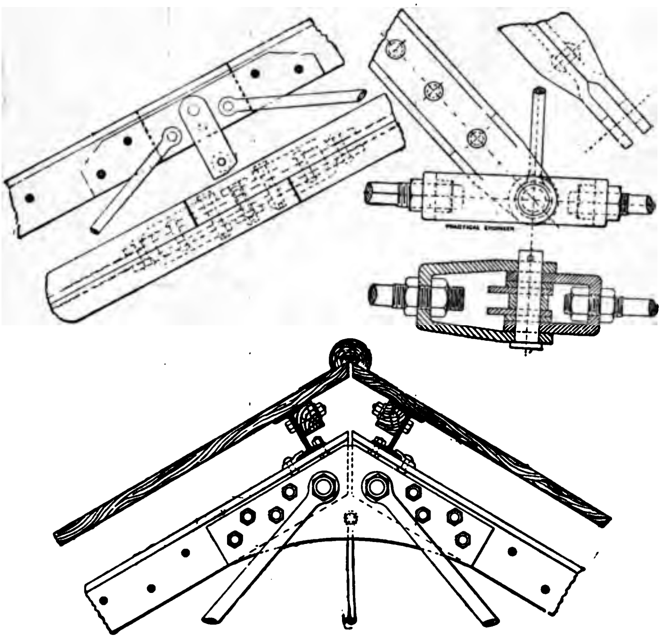
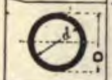
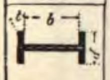


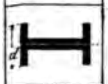
FIG. 65.

stiffen the whole roof, though this has not been shown in the figures. Some modifications of the French truss are shown in fig. 66.

In the following table all the sections may be used as struts, and hence the values of k and n are given ; while in general only some of the upper sections will be used as beams.

TABLE OF SECTIONS.

Transverse section.	Moment of inertia. I.	$z = \frac{I}{h}$	ρ^2 in $A\rho^2 = I$.	k in $A = k\rho^2$.	Average value of k.	n in $n\rho$ = depth.	Remarks.
	$\frac{\pi}{64} D^4$	$\frac{\pi}{32} D^3$	$\frac{D^2}{16}$	4π	12·5	4	
	$\frac{\pi}{64}(D^4 - d^4)$	$\frac{\pi}{32}\left(\frac{D^4 - d^4}{D}\right)$	$\frac{D^2 + d^2}{16}$	$4\pi \cdot \frac{D^2 - d^2}{D^2 + d^2}$	2·8	3·1	The values of k and n are here given on the assumption that the thickness of metal is one-tenth of the outside diameter.
	$\frac{b\,d^3}{12}$	$\frac{b\,d^2}{6}$	$\frac{d^2}{12}$	$\frac{12\,b}{d}$	..	3·5	
	$\frac{B\,D^3 - b\,d^3}{12}$	$\frac{B\,D^3 - b\,d^3}{6\,D}$	$\frac{B\,D^3 - b\,d^3}{12(B\,D - b\,d)}$	$\frac{12(B\,D - b\,d)^2}{B\,D^3 - b\,d^3}$	2·6	2·5	Values of k and n are given on the assumption that the thickness of metal is one-tenth the depth.
	$\frac{B\,D^3 - b\,d^3}{12}$	$\frac{B\,D^3 - b\,d^3}{6\,D}$	$\frac{B\,D^3 - b\,d^3}{12(B\,D - b\,d)}$	$\frac{12(B\,D - b\,d)^2}{B\,D^3 - b\,d^3}$	2·6	2·5	Values of k and n are given on the assumption that t equals one-sixteenth of the depth.
	$\frac{2\,t\,d^3 + b\,t^3}{12}$	$\frac{2\,t\,d^3 + b\,t^3}{6\,d}$	....	....	10·4	4	The value of $n=4$ is an average of a number of actual sections. The value $k = 10·4$ holds when the thickness t is one-sixteenth of the maximum dimension.
	....	....	....	....	8·8	4	The depth d is about $\frac{3}{8}\,b$ on the average.

	$\frac{(BD^2 - bd^2)^2 - 4BDbdt^2}{12(BD - bd)}$	$\frac{(BD^2 - bd^2)^2 - 4BDbdt^2}{6(BD^2 + bd^2 - 2bdD)}$	....	....	2·2	3	
	....	....	....	....	5·9	3·5	The value of n here given relates to the equation $n\rho = d$. If the equation be written $n\rho = D$, then $n = 5$.
	....	....	....	....	6	4·2	$t = \frac{D}{16}$.
	....	....	....	....	1·5	2·5	$B = \frac{2}{3} D$, and $t = \frac{D}{16}$.
	....	....	....	....	9	4·6	
	$\frac{td^3 + ht^3}{12}$	....	....	....	5·2	4·9	$t = \frac{d}{8}$.
	$\frac{bh^3}{36}$	or, $\frac{1}{24} bh^2$ / $\frac{1}{12} bh^2$	$\frac{1}{18} h^2$	....	9	4·3	The values of n and a are calculated for $b = h$.

The average value of k for some other sections will be found to be as follows :—

American Bridge Company's double tee section $k = 2{\cdot}4$
American Bridge Company's box section.. ...... $k = 1{\cdot}3$

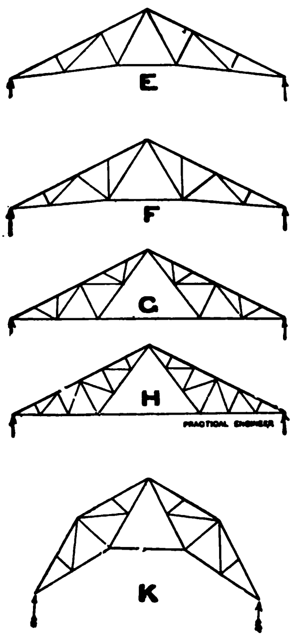


Fig. 66.

In all of these sections the radius of gyration is approximately the same about a vertical or horizontal axis.

CHAPTER XIV.

Crescent or Sickle-shaped Roof.

It is proposed to cover in a shed with a steel-framed sickle-shaped roof, the span being 50 ft. from centre to centre of bearings; the outside covering being corrugated iron of 16 B.W.G. thickness. The shed is exposed to the action of the wind in either direction. The rise of the lower chord must not be less than 8 ft. The roof is supported on walls, and anchored with holding-down bolts.

Weight of principal and purlins per square foot of area covered $= \frac{\text{span}}{25} + 4 = 6$ lb.

Snow, say .... 6 lb.

Corrugated iron, 16 B.W.G.................. 3·5 lb.

Total 15·5 lb., say 16 lb.

The principals may be spaced 10 ft. apart, this being one-fifth of the span. Then total load per principal $= 16 \times 50 \times 10 = 8{,}000$ lb.

The depth of truss at the centre is taken as one-sixth of the span, or about 8 ft., and the joints lie on the arc of a circle. Then, assuming seven bays or panels in the upper chord, and six in the lower one, with the bracing shown in fig. 67, the length of each panel in the upper chord is as nearly as possibly 9 ft. The panel load is consequently about 1,143 lb., or ·51 ton, due to the dead load only. Equal portions of each panel load will be supported at both ends, and consequently each load at the joints will be ·51 ton, with the exception of the two outside loads, which will be only half that amount. As the roof is symmetrical with regard to the centre line, the supporting forces will be equal. The line of loads d, e, f, &c., can now be set down, and the stress diagram completed for the dead load. This has been done in fig. 67. The point c in the line of loads must, of course, be in the middle of the load line, as the two supporting forces are equal. It will be noticed that the two end loads do not affect the stresses in the members of the structure. When we come to the consideration of oblique forces, such as wind pressures, then they do affect the individual members.

The wind may blow from left to right, or from right to left, and we must consider each case in turn. The slope of the panel E P, fig. 68, is 55 deg., that of F Q 37 deg., and that of S G 19 deg. Then, assuming the maximum wind pressure to be 56 lb. per square foot, the normal pressures

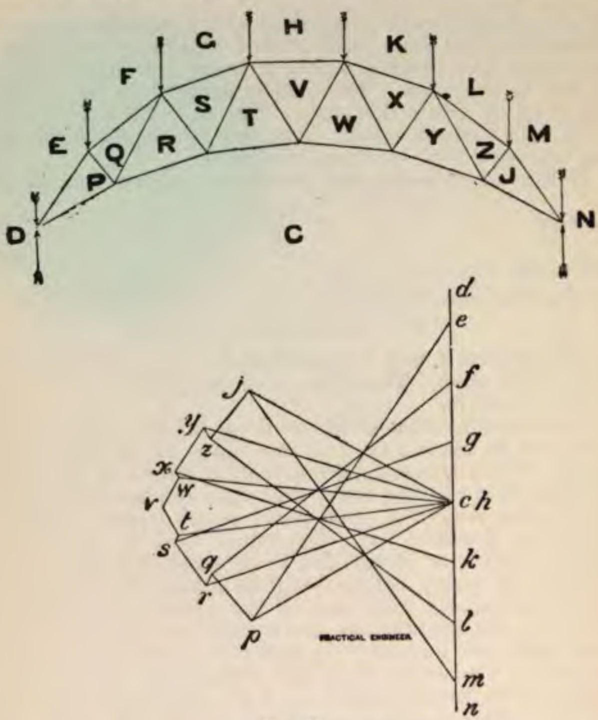


FIG. 67.

per square foot upon the above panels, obtained from fig. 60, or equation (86), are 49 lb., 39 lb., and 23 lb. respectively. The total panel loads due to wind pressure are 2 tons, 1·58 ton, and ·93 ton respectively; and after dividing them up between their joints and finding the resultants by the parallelogram of forces, we have D E = 1 ton, E F =

1·65 ton, F G = 1·16 ton, and G H = ·46 ton in the direction there shown. The supporting forces must next be found. The line of loads *d*, *e*, *f*, *g*, *h* is set down parallel and equal to the loads D E, E F, F G, and G H, fig. 66 (in the

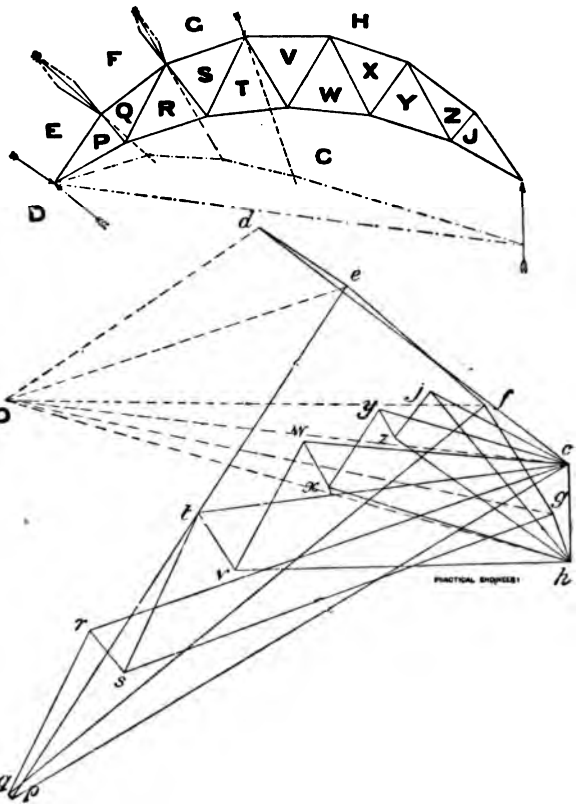


FIG. 68.—Wind from the left.

stress diagram the scale is double of that used for the loads in the upper part of the diagram). The left-hand end is assumed to be anchored down to the supporting structure,

whatever it might be, and the right-hand end is free to move horizontally; consequently the direction of the reaction there must be vertically upwards. In roofs of considerable length the free end is mounted on rollers or a sliding shoe, to allow free motion to accompany changes of temperature.

The pole O is then taken in any convenient position on either side of the load line. The *best* position for O is in a horizontal line which will about bisect the vertical height of the line loads, and far enough away from the line of loads so that the slope of the extreme radial (dotted) lines is not too great. A little experience soon shows the value of this selection. The pole O is then jointed to each of the extremities of the loads (by dotted lines). Then through each space between the loads is drawn a line parallel to the corresponding radial line in the manner previously explained with reference to fig. 18, forming in the end the funicular polygon shown dotted in the upper part of fig. 66. Thus, starting at the left-hand end of the roof, because that point is the only one yet known in the line of action of the reaction C D, the first dotted line is drawn through that point and across the space E (between the lines of action of D E and E F) parallel to the radial line O e, cutting the line of action of E F in α. Then through α and across the space F draw $\alpha \beta$ parallel to the radial line O f. Continuing this process, the line $\gamma \delta$ cuts the right-hand reaction in δ, where $\gamma \delta$ is parallel to O h. Join δ to the starting point, and through O draw O c parallel to it, cutting $c h$ in c. Join $d c$, then $c d$ and $c h$ represent the left and right hand reactions due to wind pressure. Their values are about 3·5 tons and ·87 ton respectively. The stress diagram may now be completed in the usual manner. The radial line O d does not enter into the construction at all, *because* the funicular polygon was started through the point of meeting of $d c$ and $d e$. Similarly the load D E does not affect the stresses in the individual members, but only comes in in connection with the reaction.

If both reactions are arranged to lie in the same direction—that is, parallel to one another—then the point c would lie in the line joining d to h; and $h d$ would be that direction. Then, if the funicular polygon be drawn afresh, after the reaction directions have been put in on the upper part of the figures, the then radial line O c will cut $d h$ in c, and the stress diagram may then be completed. This state of affairs just described approximates to that of both ends of roof fixed. In fig. 68 the stress diagram is drawn on the

supposition that the right-hand end of the truss is free, and is to double the scale to which the loads are drawn in the upper part of the figure.

The wind-pressure loads being now transferred from the left to the right, the corresponding stress diagram, fig. 69, is obtained. This is again drawn to a different scale. The corresponding stresses in the three diagrams are tabulated on page 158.

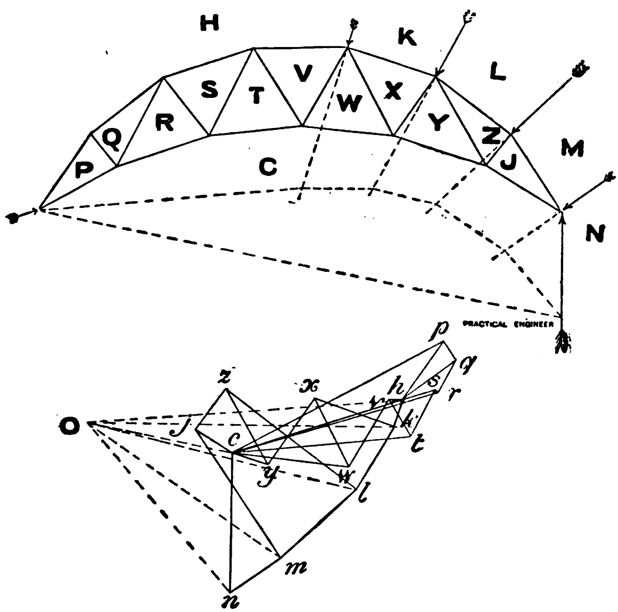


FIG. 69.—Wind from the right.

The fifth and sixth columns of this table have been obtained by adding the dead load stress, column 2, to one of the wind-pressure stresses, the maximum sum being placed in column 5, and the minimum in column 6. In the latter it will be noticed that nearly all of the minimum stresses are compressive, while many of the maximum stresses are tensile, thus showing that some of the members will be sometimes in tension and at other times in compression

depending upon the direction of the wind. This necessitates

Member.	Dead load. Total stress. Tons.	Total stress due to wind from the left. Tons.	Total stress due to wind from the right. Tons.	Maximum total stress. Tons.	Minimum total stress. Tons.
E P	− 3·05	− 5·5	+ 1·2	− 8·55	− 1·85
C P	+ 1·97	+ 5·8	− 4·1	+ 7 77	− 2·13
P Q	+ 0·52	+ 0·04	− 0·4	+ 0·56	+ 0·12
F Q	− 2·6	− 5·5	+ 1·16	− 8·1	− 1·44
Q R	− 0·11	+ 1·77	− 0 63	+ 1·66	− 0·74
C R	+ 2·2	+ 4 5	− 3·65	+ 6·7	− 1·45
R S	+ 0·45	− 0·55	+ 0 06	+ 0·51	− 0·1
G S	− 2·5	− 4·1	+ 0·63	− 6·6	− 1·87
S T	+ 0·06	+ 1·7	− 0·9	+ 1·76	− 0·84
C T	+ 2·35	+ 3·3	− 3·05	+ 5·65	− 0·7
T V	+ 0·27	− 0·67	+ 0·76	+ 1·03	− 0·4
H V	− 2 5	− 3 0	− 0·22	− 5·5	− 2·72
V W	+ 0·27	+ 1·4	− 1·34	+ 1·67	− 1·07
C W	+ 2·35	+ 2·35	− 2·0	+ 4·7	− 0·35
W X	+ 0·06	− 0·5	+ 1·34	+ 1·4	− 0 44
K X	− 2·5	− 2·25	− 1·43	− 4·75	− 3·93
X Y	+ 0·45	+ 0·9	− 1·38	+ 1·35	− 0 83
C Y	+ 2·2	+ 1·75	− 0·63	+ 3·95	− 1 57
Y Z	− 0·11	− 0·31	+ 1·52	+ 1·41	− 0·42
L Z	− 2·6	+ 1·98	− 2·82	− 5·42	− 0·62
Z J	+ 0·52	+ 0·56	− 0·9	+ 1·08	− 0 38
M J	− 3·05	− 1·99	− 2·64	− 5·69	− 5·04
C J	+ 1·97	+ 1·4	+ 0·76	+ 3·37	2·73
C D	1·8	3 5	2·8	—	—
C N	1·8	0 87	2·2	—	—

the designing of such members both as ties and struts, and

if any of them are located in the lower chord, the chord must be braced laterally to maintain it in the plane of the truss. Now, the span of the roof is 50 ft., and the expansion due to fluctuation of temperature of the whole truss is not more than about $\frac{3}{16}$ in. on either side of its mean position, and consequently it is the general practice to fix both ends of the truss to the supporting structure, and then the change of kind of stress in the lower chord does not take place. We may then proceed to design all the members in the left half of the roof as if the wind blew from the left and the truss were fixed at the left end. The stresses in the right half members will be generally small compared with those on the left, when the wind is blowing from the left. In the same way the members in the right half of the roof may be designed with the wind blowing from the right, and the truss fixed at the right-hand end. The maximum stresses in homologous members must then be the same; for instance, the maximum stress in P Q is the same as the maximum stress in J Z. With this assumption, the maximum stresses are—

Members.	Stress in tons.	Members.	Stress in tons.
E P and M J	− 8·55	C T and C W	+ 5·65
C P and C J	+ 7·77	H V	− 5 5
P Q and Z J	+ 1·08	Q R and Y Z	+ 1·66 or − ·42
F Q and L Z	− 8·1	R S and X Y	+ ·51 or − ·1
C R and C Y	+ 6·7	S T and W X	+ 1·76 or − ·44
G S and K X	− 6·6	T V and V W	+ 1·67 or − ·4

The last eight members in the table are sometimes in tension and sometimes in compression, and therefore will have to be designed as both struts and ties. The stress (varying) in these members may be taken as 8 tons per square inch. A very convenient form for these members will be small channels, in which the average value of k [in equation (84)] is about 7. The sectional area of the strut thus obtained for a load of half a ton is about 1 square inch. The sectional area of a channel steel $2\frac{3}{8}$ in. by $1\frac{1}{16}$ in. by $\frac{1}{4}$ in. is slightly more than 1 square inch; therefore this section will do. It gives ample strength in tension also. The ties can be conveniently made of flat bars, of which C P and

C J may have a section 2 in. by ½ in., P Q and Z J one rod 2 in. by ¼ in., C R and C Y one rod 2 in. by $\frac{7}{16}$ in. or 2 in. by ½ in., and T C and W C one rod 2 in. by ⅜ in. The most heavily stressed strut is E P or J M. These are conveniently

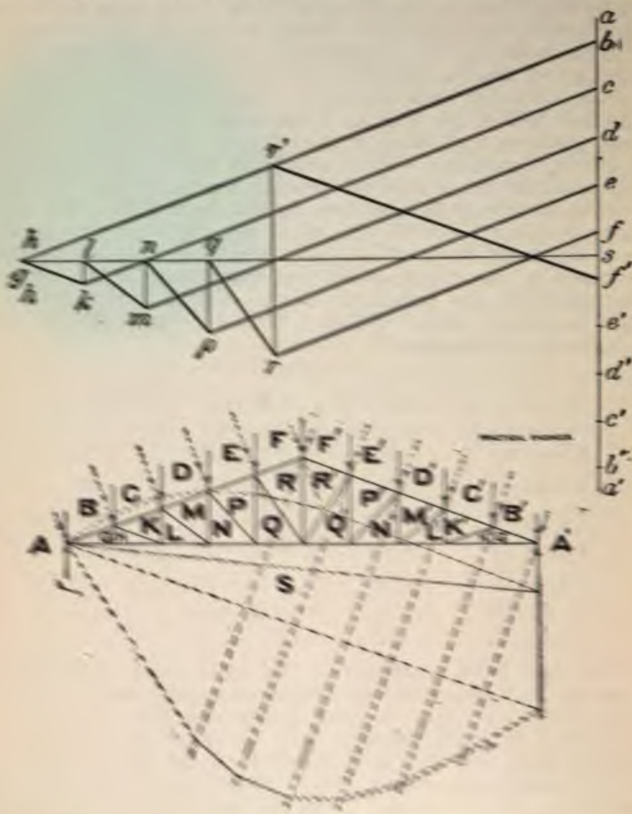


Fig. 78.

made of T steel, if the section required is not too great. The lower end of either strut is fixed, and hence it may be treated as a strut fixed at one end and round at the other.

Using 8 tons per square inch as the working stress, and putting in the values of the other constants in equation (84), we obtain 4 square inches as the required sectional area. The T 6 in. by 3 in. by $\frac{1}{2}$ in. will be suitable. The whole upper chord may be made in two or three pieces, having bends at the joints. Perhaps the most convenient form would be a rivet joint in the middle of each of the members G S and K X (see fig. 70). The corrugated iron will be bolted direct to the purlins, which may be of channel tie or angle section, and so spaced that the desired length of plate may be conveniently bolted to the purlins.

CHAPTER XV.

King Rod Truss.

It is required to design an iron "king rod truss," of 75 ft. span, the purlins to be of wood, and the covering to be slate, with wood lining. The ties to be round iron rod, and pin joints throughout. The main rafter of each truss to be made of tee iron, or two angles riveted together, and the inclined struts to be made up of a pair of channel irons, riveted together, back to back. The slope of roof to be 20 deg.

Weights per square foot of area covered—

Slate	8 lb.
Wood lining, $\frac{3}{4}$ in. thick	3 lb.
Snow	6 lb.
Truss and purlins $= \frac{\text{span}}{25} + 4 =$	7 lb.
Total	24 lb. per sq. ft.

Distance apart of principal trusses—say 12 ft. Dead weight of one truss and covering

$$= \frac{12 \times 75 \times 24}{2240} = 9{\cdot}6, \text{ or say 10 tons.}$$

And if the truss be divided into ten panels, the panel load is 1 ton. The skeleton of the truss is given in fig. 70, together with the dead-load stress diagram for one-half the truss; the stresses in the other half being the same. The individual stresses are collected in the following table. It

will be noticed that the stress in G H and G_1 H_1 is zero, but that member is still retained in the structure for the sake of supporting the heavy tie rods S H and S_1 H_1, and preventing them from sagging. The vertical members are ties, and the inclined members are struts. The stresses in these members are not changed by the imposition of the wind pressures. The left end is assumed to be bolted down to the supporting structure, while the right-hand end rests upon rollers, or other arrangements for providing a free end.

The several stresses may be tabulated as follow:—

Member.	Dead load. Tons.	Wind from left. Tons.	Wind from right. Tons.	Maximum stress.	Diameter of tie rods in inches.	Length of struts in inches.
S H	12·2	9·2	1·9	21·4	2	..
S H_1	12·2	3·6	7·7	19·9	2	..
S L	10·8	8·0	1·9	18·8	1⅞	..
S L_1	10·8	3·6	6·2	16·8	1¾	..
S N	9·5	6·5	1·9	16·0	1¾	..
S N_1	9·5	3·6	4·7	14·2	1¾	..
S Q	8·2	5·1	1·9	13·3	1⅝	..
S Q_1	8·2	3·6	3·3	11·8	1⅝	..
B H	13·0	8·4	3·8	21·4	..	96
B_1 H_1	13·0	3·8	8·4	21·4	..	96
C K	11·7	7·2	3·8	18·9	..	96
C_1 K_1	11·7	3·8	7·2	18·9	..	96
D M	10·1	6·0	3·8	16·1	..	96
D_1 M_1	10·1	3·8	6·0	16·1	..	96
E P	8·7	4·8	3·8	13·5	..	96
E_1 P_1	8·7	3·8	4·8	13·5	..	96
F R	7·2	3·6	3·8	11·0	..	96
F_1 R_1	7·2	3·8	3·6	11·0	..	96
G H	0	0	0	0	½	..
G_1 H_1	0	0	0	0	½	..
K L	·5	·5	0	1	½	..
K_1 L_1	·5	0	·5	1	½	..
M N	1·0	1·0	0	2	⅝	..
M_1 N_1	1·0	0	1·0	2	⅝	..
P Q	1·5	1·5	0	3	¾	..
P_1 Q_1	1·5	0	1·5	3	¾	..
R R_1	4·0	2·0	2	6	1⅛	..
H K	1·4	1·5	0	2·9	..	96
H_1 K_1	1·4	0	1·5	2·9	..	96
L M	1·7	1·8	0	3·5	..	112
L_1 M_1	1·7	0	1·8	3·5	..	112
N P	2·0	2·2	0	4·2	..	136
N_1 P_1	2·0	0	2·2	4·2	..	136
Q R	2·3	2·6	0	4·9	..	162
Q_1 R_1	2·3	0	2·6	4·9	..	162

The wind pressure from fig. 62 is about 24 lb. to the square foot normal to the surface. Total wind pressure on one rafter

$$= \frac{24 \times 12 \times 40}{2240} = 5{\cdot}1 \text{ tons};$$

or say 1 ton per panel, where 40 ft. is the length of the rafter. The stress diagram for wind pressure from either

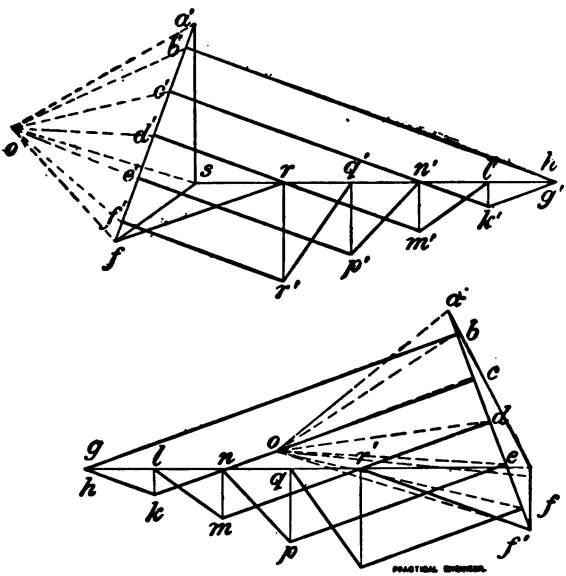


FIG. 71.

side will be found in fig. 71, the upper figure representing the wind from the right on free end, and the lower figure the wind from the left on fixed end. The necessary funicular polygons are shown in fig. 70 (dotted), the upper one being in connection with the lower stress diagram, and *vice versa*. A peculiar feature in the wind stress diagrams is that the point *h* is also the points *k*, *l*, *m*, *n*, *p*, and *q*, which indicates that there will be no stress in the vertical and diagonal members of one-half the truss due to wind pressure

alone. Thus, when the wind blows from the left, there will be an absence of stress in the right half of the bracing, due to wind pressure, and *vice versa*. Great care should be exercised in finding the point s, otherwise the stress diagram will not close.

The diagonals are composed of a pair of channel irons, riveted back to back. From the table of values of k, its value for this section will be found to be 9. The value of the constant c for wrought-iron struts with round ends is $\frac{1}{2250}$, and the stress allowed per square inch is 7 tons. Substituting in equation (73), we obtain the following results :—

Member.	Sectional area in square inches.	Composition.
H K and $H_1 K_1$	2·75	Two channels, each $3\frac{1}{2} \times 1\frac{1}{2} \times \frac{3}{8}$ inch.
L M and $L_1 M_1$	3·4	Two channels, each $3\frac{1}{2} \times 1\frac{1}{2} \times \frac{1}{2}$ inch.
N P and $N_1 P_1$	4·5	Two channels, each $4\frac{1}{2} \times 2 \times \frac{3}{8}$ inch.
Q R and $Q_1 R_1$	5·8	Two channels, each $4\frac{1}{2} \times 2 \times \frac{1}{2}$ inch.

The principal rafter may be made up of a pair of angle irons riveted together in the form of a tee. A single tee iron will probably not be strong enough, using the commercial sizes. The rafter cannot bend in the plane of the roof covering on account of the purlins being so many, and rigidly secured to the rafter. The value of k (from table) when bending takes place in the plane of the truss is 2·65. Then allowing 6 tons per square inch for the stress, equation (73) gives a sectional area of 6·7 square inches, assuming the ends round and the length of the strut equal to the length of one panel. There must be a considerable margin of safety on this assumption, because the conditions in this instance do not even approach those of round ends. Two angles, each $6\frac{1}{2} \times 3 \times \frac{3}{8}$, give a combined sectional area of 6·75 square inches, which will be ample.

The purlins may be of deal, spaced, say, 3 ft. apart. Total load sustained by one purlin $= 12 \times 3 \times (17 + 24)$ lb., uniformly distributed. The maximum bending moment in pound-inches due to this load is $\frac{144 \times 1476}{8}$, which equals

$\frac{fI}{h}$. The section is rectangular, and hence this latter quantity becomes $\frac{1}{6} f b d^2$ where b = breadth and d = depth. Inserting ·75 ton for f, we get—

$$95 = b d^2$$

If $b = 2$, $d = 6·9$; and if $b = 2\frac{1}{2}$, $d = 6·1$. A purlin $6 \times 2\frac{1}{2}$ inches will be suitable.

A detail of a suitable shoe is given in fig. 64 in plan and elevation, together with a detail of one of the joints in the lower chord. Such a joint is very convenient to make and fit up. A modification of the above truss is that shown in fig. 72, sometimes called a queen rod truss. The struts are vertical, and the tie rods are inclined in the opposite direction to the diagonal members in the king rod truss. In fig. 72, the lines representing struts have been thickened.

FIG. 72.

It is sometimes useful to check by arithmetical calculation the results obtained by graphical processes; and this, in many cases, can be easily done by Ritter's "Method of Sections." The process consists of cutting some of the members of a frame and replacing the parts removed by the forces exerted by the removed parts upon the remaining parts of the structure.

In fig. 73 the letters denote both the length and the name of the individual members; and let A denote the point of application of the supporting force R. Also let $\alpha \beta \gamma$ denote the upper ends of the vertical posts, while $\theta \phi \pi$ denote the lower ends of the same posts. Let the load at each joint in the upper chord be w. Now cut the truss by the line $m_1 m_1$, fig. 72 (x), and put in the forces at the several sections with which the removed parts acted upon those which are left; and let the total stress in each member be denoted by f, with the letter of that member as suffix. For example, the stress in the member a will be denoted

by f_a, and so on. Taking moments about a, we obtain from the second law of equilibrium—

$$R \times e - f_e \times k = o,$$

and as $$R = 3{\cdot}5\ w, f_e = R \cdot \frac{e}{k} = \frac{3{\cdot}5\ we}{k}.$$

Taking moments about A, we get—

$$f_k \times e - w \times e = o, \text{ or } f_k = w.$$

Taking moments about θ, and denoting the length of the perpendicular from θ on to a by p,

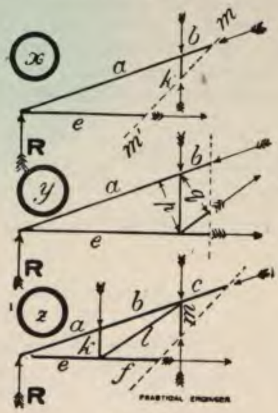


FIG. 73.

$$f_b \times p - R \times e = o, \text{ and } f_b = \frac{3{\cdot}5\ we}{p}.$$

To get f_a, cut the truss by the line $m_2\ n_2$, fig. 72, and take moments about θ. Then—

$$f_a \times p - R \times e = o, \text{ or } f_a = \frac{3{\cdot}5\ we}{p}.$$

Now cut the truss with the line $m_3\ n_3$, fig. 72 and fig. 73 (z), and take moments about A. Then—

$$f_l \times 2q - we = o, \text{ or } f_l = \frac{we}{2q}.$$

Also by taking moments about a, we get—

$$f_l \times q + f_f \times k - \mathrm{R} \times e = o; \text{ that is,}$$

$$f_f = \frac{\mathrm{R}e - qf_l}{k} = \frac{3{\cdot}5\,we - {\cdot}5\,we}{k} = \frac{3\,we}{k}.$$

Proceeding in like manner to cut the truss by the line $m_4 n_4$, fig. 72, we obtain by taking moments about A—

$$w \times e + w \times 2e - f_m \times 2e = o, \text{ or } f_m = \frac{3\,w}{2},$$

and by taking moments about ϕ, we get—

$$f_c \times 2p + w \times e - \mathrm{R} \times 2e = o;$$

that is, $$f_c = \frac{e}{2p}(7\,w - w) = \frac{3\,we}{p}.$$

Proceeding in this way, we obtain the following stresses in the several members of one-half of the stress :—

Member.	Stress.	Member.	Stress.	Member.	Stress.
a	$\frac{3{\cdot}5\,we}{p}$	f	$\frac{3\,we}{k}$	r	$2\,w$
b	$\frac{3{\cdot}5\,we}{p}$	g	$\frac{2{\cdot}5\,we}{k}$	l	$\frac{{\cdot}5\,we}{q_1}$
c	$\frac{3\,we}{p}$	h	$\frac{2\,we}{k}$	n	$\frac{we}{q_2}$
d	$\frac{2{\cdot}5\,we}{p}$	k	w	..	..
e	$\frac{3{\cdot}5\,we}{a}$	m	$1{\cdot}5\,w$	s	$\frac{1{\cdot}5\,we}{q_3}$

It must be evident that this method is more easily applied to a bridge with parallel booms and open bracing.

CHAPTER XVI.

VENTILATORS.

THUS far no mention has been made of the action of the wind on ventilators, because, in general, a ventilator contains more members than are necessary for statical equilibrium. These extra members are called *redundant* members. There are two methods of dealing with the

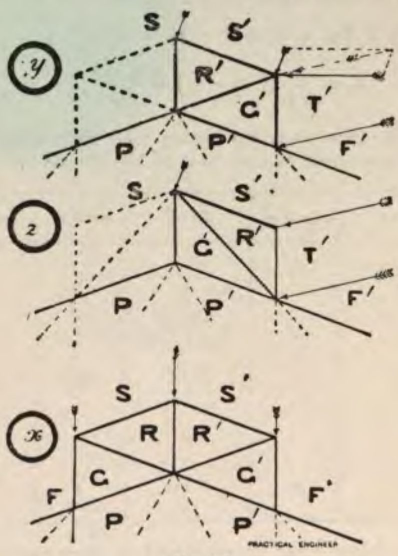


FIG. 74.

stresses in these redundant members. The accurate method is to find the stress in each of the members, by considering the whole of them together, and then applying the dynamical principle known as that of "least work." Except in the simple cases this method is exceedingly laborious, and, of course, would not be used in such a

trivial affair as that of an ordinary ventilator. The other method is to cut out the redundant members, and find the statical stresses in those that remain. This will here be adopted. The load on the ventilator is small, and were it not for gusts of wind the original framework would probably be ample. Bracing is then introduced to prevent deformation under wind pressure. The skeleton of a ventilator frame is shown at *x*, fig. 74, subjected to the dead load only. The different portions of the dead load are conveyed to the roof truss by the three vertical posts of the ventilator. The bracing may be arranged as at *y* or *z*, fig. 74. With the wind from the right the parallelogram forming the right half of ventilator (omitting the brace R_1 G_1) will have a tendency to be flattened out, and to prevent this a diagonal member may be introduced, as in *z* or as in *y*; the former in compression, and the latter in tension. The magnitude and direction of the wind pressures are obtained by finding the wind pressures normal to the sloping portion of the roof and the wind pressure (horizontal) on the vertical side of ventilator G_1 T_1, and dividing it equally between the upper and lower ends of the vertical post. We then have at these points a pair of forces which can be compounded by the parallelogram of forces, as shown at the joint T_1 G_1 R_1 S_1, in fig. 74 (*z*). The resultants only are shown at *y*, fig. 74. The whole of the wind pressures from the right can be statically sustained by the right half of the ventilator, the members F G, G R, and R S not being required, and can therefore be omitted when considering the stresses due to wind from the right. The stress diagram can then be drawn in the usual manner.

CHAPTER XVII.

KNEE BRACING.

When roof trusses are supported by columns, the effect of wind pressure on the roof covering and on the side of the structure is to tend to deform it, as shown by dotted lines at C, fig. 75. This is more especially the case when the columns are assumed to be hinged or round at their ends, though the action is similar, but less in degree, when the lower ends are assumed to be rigidly fixed. To prevent

this deformation the knee braces at D, fig. 75, are inserted. The original truss will act as a rigid body, and can be treated as such when finding the stress in the braces. If the stress diagram is drawn for the dead loads only, or for any *vertical* components of loads, it will be found that no stress is produced in the knee braces; consequently, it is only the horizontal components of the various loads which have any effect in straining the knee braces. We may then replace the main truss by a single beam, as shown at E, fig. 76. The truss may be assumed to be hinged to the supporting columns, otherwise the principle of "least work" must be used; and this for the present we wish to avoid. The column may be assumed to be fixed at the lower extremity if its base is large, and well fastened to a deep and rigid foundation; but as this condition is sometimes far from being satisfied, we have taken the simplest case, and the case of maximum stress, namely, that of round-ended columns. At E, fig. 76, there is shown

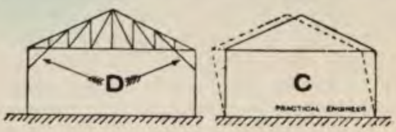


FIG. 75.

a couple of braces, but it will be seen that one alone is sufficient for static equilibrium by referring to the diagram F, where the left brace is removed, and the left column merely supports the left-hand end of the horizontal beam. A horizontal force is applied in a direction from right to left at the top of the column, which is resisted horizontally by the equal and opposite reaction at the foot of the column. It is clear the left column is unable to support any of the horizontally impressed force, and that the angle between the cross beam and the right column has a tendency to increase under the action of the impressed force; the increase being resisted by the knee brace. The several forces on the cross beam and right column are represented by arrows at G, fig. 76. If the impressed force acted in the opposite direction, the whole of the forces shown at G would be reversed, and the knee brace would be a strut instead of a tie.

Our principal object in the present investigation is to determine the stresses in the truss members, with the knee

brace introduced. To accomplish this end with greater clearness, the cross beam has been separated from the right-hand column in the diagram H, fig. 76, and the whole of the forces acting *on* each of them have been put in, the stress in the knee brace having been resolved into its components in the horizontal and vertical directions These components will be represented by f_h and f_v respectively. The column is divided by the knee-brace joint into two parts whose lengths are α and β, while the cross beam is divided in the same manner into the lengths θ and ϕ. Considering the equilibrium of the column first, and denoting the horizontally impressed force by h, and consequently the reaction at the foot of the column by h, then

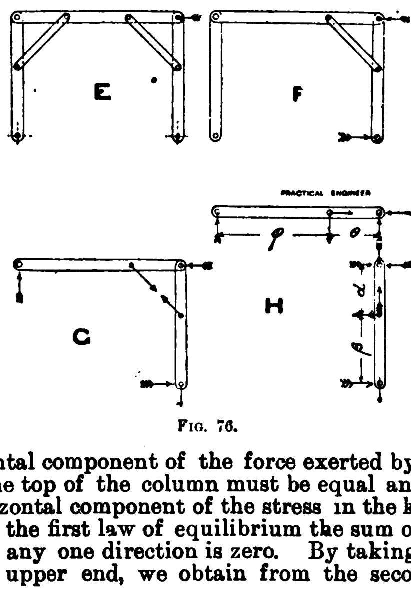


FIG. 76.

the horizontal component of the force exerted by the cross beam on the top of the column must be equal and opposite to the horizontal component of the stress in the knee brace, because by the first law of equilibrium the sum of the components in any one direction is zero. By taking moments about the upper end, we obtain from the second law of equilibrium

$$f_h = h\left(1 + \frac{\beta}{\alpha}\right)$$

Referring to diagram F or G, fig. 76, representing the reaction of the left-hand column by R, we have, by taking

moments about the foot of the right-hand column, the relation

$$R = h\frac{x + y}{y + a},$$

and then, considering the equilibrium of the cross beam, we obtain, by taking moments about the right-hand end (diagram B),

$$f_1 = h\frac{x + y}{a},$$

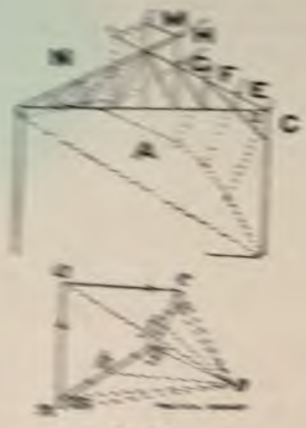


FIG. 77.

while the vertical component of the reaction of the column on the right-hand end

$$= f_1 - R = \frac{h\,a}{a}\left(\frac{x + y}{y + a}\right)$$

There will now be no difficulty in drawing the stress diagram for the whole truss when one knee brace is used at a time. The total reaction at the free end will equal the reaction due to vertical loads alone *plus* the vertical reaction R, due to the horizontal loads. The vertical reaction at the braced end will equal the reaction due to vertical loads *plus* the vertical pressure of the braced column on the cross beam or truss, due to the horizontally impressed force h. The latter quantity is $f_1 - R$. The magnitude and direction of

the resultant reaction at the braced end of beam or truss, will be represented by the diagonal of the rectangle, whose adjacent sides are $f_h - h$ (horizontal), and (vertical) the reaction due to dead loads *plus* the vertical reaction f_v – R, due to the horizontal load h.

The stresses in a structure calculated as above would actually exist if the knee braces were designed to resist tension only; but, generally speaking, they are as well fitted

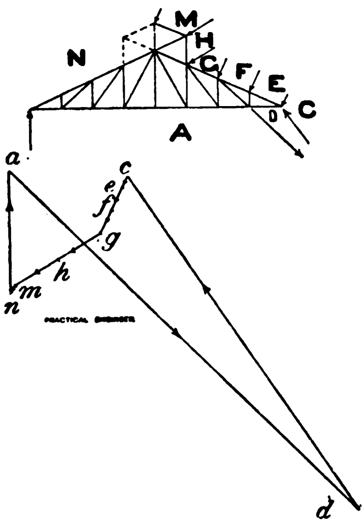


FIG. 78.

to resist compression as tension, and so make a stiffer structure.

At first sight, under these conditions, it would appear that it would not cause great error to divide the loads equally between the two columns; but upon a little consideration it must be manifest that this will only hold when all the loads are symmetrical with respect to the centre of the truss, and that a wind pressure on one side of the roof only will cause a greater reaction on the end adjacent to the wind, and consequently a greater stress in the knee brace.

The method of obtaining the line of loads from which to draw the stress diagram of the truss members is shown in figs. 77 and 78. For the sake of convenience the wind pressures are assumed to be found, and the column is hinged at its ends, while the knee brace is in tension. The structure as a whole is in equilibrium under the action of the wind pressures and the reactions N A and A C, fig. 77, the dead loads being omitted. The reactions must first be determined. Set down the line of loads *c n*. Take any pole O and draw the funicular polygon (shown dotted in the upper part of figure). The closing line being drawn, its radial companion O *a* is then put in, cutting the vertical reaction *n a* in *a*. Then *a c* represents the reaction at the foot of the right-hand column. It will be noticed that the funicular polygon

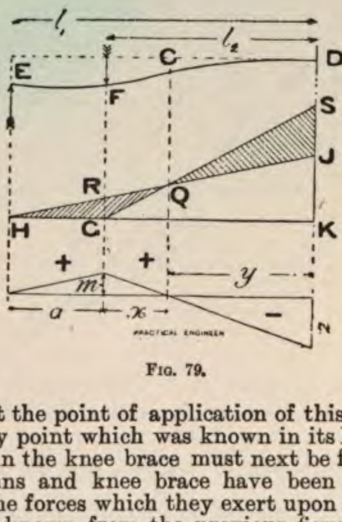


FIG. 79.

was started at the point of application of this reaction, that being the only point which was known in its line of action.*

The stress in the knee brace must next be found. In fig. 78 the columns and knee brace have been removed and replaced by the forces which they exert upon the truss. Of these N A is known from the previous figure. The stress A D in the knee brace is most easily found by taking moments of the forces on the column round the upper end. In this way the reaction A C (fig. 77), multiplied by the length of the column, equals the stress in the knee brace,

* For detailed description of the method of finding the supporting forces, refer to the Warren girder, page 39, and the unsymmetrical structure, page 40, *ante*.

multiplied by the perpendicular distance of the upper end of column from the knee brace. Putting in the numerical quantities, we obtain the stress in A D, fig. 78. The line $a\,d$ is then drawn parallel to A D from a, and of a length equal to that just found above. The point d is thus secured. Join $d\,c$, then $d\,c$ is the reaction of the end of the column on the truss, and the line of loads is completed, so that the stress diagram may be drawn.

Should the column be *fixed* at its base—*i.e.*, securely fastened to a good foundation by holding-down bolts—the corresponding reactions, &c., may be approximately calculated. Although the column may bend considerably, the upper end and the point of attachment of the knee brace still remain approximately in the same vertical line; or, in other words, the deflection of these points will be the same. This is the key to the solution. In fig. 79, let E F C D represent the column fixed at D. The bending moment diagram (and consequently the stress diagram if the column is parallel) due to each load will be a triangle. Let G S K be the diagram due to the load at F, while H J K is the diagram due to the load at E. These two loads tend to bend the column in opposite directions, and consequently the resultant stress diagram will be formed by taking the difference of the ordinates of the two triangles. This is the same, as the vertical intercepts of the shaded portion, in which the intercepts of the triangle H Q G are, say, positive, and those of the triangle S T Q negative. Transferring the shaded area by pure shear to the lower part of the figure is the same as plotting a new diagram on a horizontal line as base. The triangle H Q G may, for convenience, be divided into the two triangles H R G and R Q G.

By the graphic theory of deflection (previously given), the deflection of the point E, measured from the tangent at D, equals the moment of the area of the triangle H Q G about H E, *minus* the moment of the area of the triangle S J Q about H E. This is the same as—

Moment of H R G + moment of R Q G − moment of S J Q about H E, which equals

Area H R G × the distance of its centre of gravity from H E
+ area R Q G × " " " "
− area S J Q × " " " "

Let δ_1 be this deflection, then

$$\delta_1 = \frac{a\,m}{2} \times \frac{2\,a}{3} + \frac{x\,m}{2}\left(\frac{x}{3} + a\right) - \frac{y\,z}{2}\left(\frac{2\,y}{3} + x + a\right)$$

and if δ_2 be the deflection of the point F from the same tangent at D, then, similarly,

$$\delta_2 = \frac{m\,x}{2} \times \frac{x}{3} - \frac{y\,z}{2}\left(\frac{2\,y}{3} + x\right).$$

Subtracting the latter equation from the former, there remains

$$o = \frac{a^2\,m}{3} + \frac{a\,m\,x}{2} - \frac{a\,y\,z}{2},$$

because the deflection of E and F are the same.

But from similar triangles

$$\frac{z}{y} = \frac{m}{x},$$

and replacing z in the above equation by its equivalent $\frac{m\,y}{x}$ we obtain

$$2\,a\,x + 3x^2 = 3\,y^2,$$

and after substituting $l_1 - l_2$ for a, and $l_2 - y$ for x, we find that

$$y = \frac{2\,l_1\,l_2 + l_2{}^2}{4\,l_2 + 2\,l_1}.$$

In this way the point Q, and consequently the point C, is obtained without knowing the magnitude of the forces at F and E. This point C is a point of zero stress, and consequently a point of inflexion; in other words, the portion E C of the column behaves exactly like a beam of that length supported at E and C. Then, in treating a braced structure with fixed columns, it is necessary to find the point of inflexion C in the columns, and treat that part of the column between the point of inflexion and the upper end as a column with round ends.

On reference to fig. 77, it will be noticed that the truss is exceedingly shallow at the section where great strength is required; and this absence of depth produces excessive stress in the members located in that region. Should the roof be very large, and so situated that it cannot be divided up into a number of smaller trusses, it is customary to use a form similar to that shown in fig. 80.

Here it will be noticed that the truss is deepest at the section where the knee bracing occurred in the previous roof; consequently the stresses are more nearly uniform throughout the truss. The roof principal here shown is

formed of a pair of girder-shaped rafters, hinged together at the vertex, and resting upon hinge or cup shaped joints; and it is evidently well suited to accommodate itself to great changes of temperature. The bearings at the base are well bedded in good foundations, or are tied together under

FIG. 80.

the floor of the building by iron rods. The principal is in reality a three-hinged arch, and will be included in the investigation of arches in general. This form of roof is much used for very large railway stations.

CHAPTER XVIII.

THE QUADRANGULAR TRUSS.

THIS form of truss (fig. 81) is also sometimes used for large spans, notably the station roof of the Central Railway of New Jersey, at Jersey City, U.S.A. That shown at E, in the figure, has "queen rod" bracing, while that shown at F has "king rod" bracing. In the former, the majority of the diagonals are struts, and in the latter, ties. If the stress diagram is drawn for the two forms of bracing, it will be found that the stress in the braces decreases from the extreme ends to some section between the ends and vertex, there changes sign, and then increases towards the vertex. We shall thus have a certain number of braces in tension, and the remainder in compression. It is often convenient to maintain all of one kind of bracing (say the diagonals) in the same state of stress, and this may be done by reversing

the direction of the diagonal bracing at and after the section where the stress changes sign. Thus, in form E (fig. 81), the two verticals nearest the vertex are in tension, the remainder being in compression; and, similarly, the corresponding diagonals are in compression, the remainder being in tension. To maintain all the verticals as struts and the diagonals as ties, all that is required is to reverse the direction of the diagonals whose stress sign is different from those near the extremities, as shown in the left half of G.

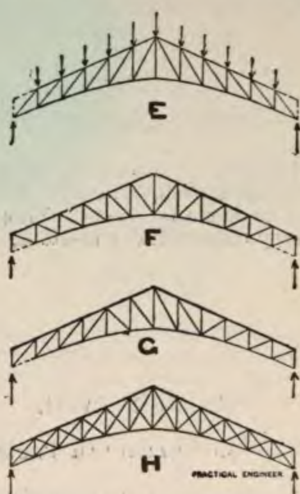


Fig. 81.

In some instances the two forms of bracing are used together, as shown at H (fig. 81), the stresses in which are statically indeterminate, and can only be accurately found by the method of least work. An assumption often made in practice is that each system of bracing sustains half the load, and the vertical posts distribute the load equally between the two systems.

It may be interesting to inquire into the reason for the change in kind of stress in the bracing, and for that purpose it will be more conclusive to resort to the method of sections. A few panels are shown drawn to a larger scale in fig. 82, in

which R is the left-hand supporting force, and a load of w imposed at each joint in the top member. The horizontal distance apart of the vertical posts may be denoted by x, and the stress in any member by f, with the suffix t for the top member of a panel, and the suffixes d and b for the diagonal and bottom member of the panel respectively. Now cut the truss through the nth panel—in this case the fifth—and substitute for the removed parts the stresses exerted by them on the remainder of the truss. These are shown in the diagram. Let there be N panels in the truss altogether. Taking moments about P, we get—

$$f_d \times d + w \, . \, x + 2\,w\,x + 3\,w\,x + \ldots . (n-1)\,w\,x - f_b\,a = 0;$$

or, $$f_d d + w\,x\,(1 + 2 + 3 + \ldots n - 1) - f_b\,a = 0;$$

or, $$f_d = \frac{a}{d} \, . \, f_b - \frac{w\,x\,n}{2\,d}\,(n - 1).$$

Again, taking moments about N,

$$\mathrm{R}\,x\,(n - 1) - \frac{(n - 2)(n - 1)}{2}\,w\,x - f_b\,p_n = 0;$$

and $$\mathrm{R} = \left(\frac{\mathrm{N} - 1}{2}\right) w;$$

therefore, $$f_b = \frac{w\,x\,(n - 1)}{2\,p_n}\,[\mathrm{N} - n + 1].$$

Substituting this in the equation for f_d, we have—

$$f_d = \frac{w\,x\,(n - 1)}{2\,d}\left[\frac{a}{p_n}\,(\mathrm{N} + 1 - n) - n\right].$$

If the booms were parallel, then a would be equal to p_n, and d would be some multiple of $(n - 1)$, say $r\,(n - 1)$, and the stress in the nth diagonal would then be

$$\frac{w\,x}{2\,r}\,(\mathrm{N} + 1 - 2\,n).$$

This can never be negative, and decreases as n increases, so that the stresses in the diagonals would be in arithmetical progression, and would decrease from the extremities to the centre of the truss. But with the given truss in fig. 82, d is some function of $(n - 1)$—very nearly a simple multiple of

it—and a decreases while p_n increases from the extremities to the centre; hence the stress in the diagonals will decrease much more rapidly from the ends to the centre of the truss (fig. 81) than with parallel booms, and consequently will pass through zero, and change sign somewhere between the ends and the centre of the truss. The same holds for the verticals, for

$$f_v = \frac{w}{2}\left[\frac{a}{p_n}(N + 1 - n) - (n - 2)\right]$$

which is very similar to the last equation.

Another form of roof truss, in which a kind of knee bracing is introduced, is that shown in fig. 83. The lower boom of the truss often assumes the shape of a circle or an

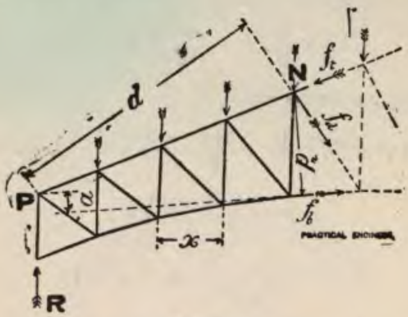


FIG. 82.

ellipse. The truss is more especially used when a number of bays run side by side, so that one pillar supports the ends of two trusses, as shown in the upper part of the figure. It is a light and economical form of roof when used as shown. Considering a single pillar, it is evident from symmetry that one truss acts upon its companion adjoining it, horizontally; the forces being D E and E F for that shown on the left of the enlarged view. In reality the whole truss is made up of three different trusses. Looking at the view of one column and a portion of two trusses shown in fine lines (lowest sketch), it is evident the first two panels on either side of the pillar form a cantilever, its extremity in the middle view of the figure being the hinge where the force H J is applied.

*The three panels on either side of the vertex of the truss form a truss of themselves, similar to the ordinary roof truss. It will be easier to find the forces D E and E F

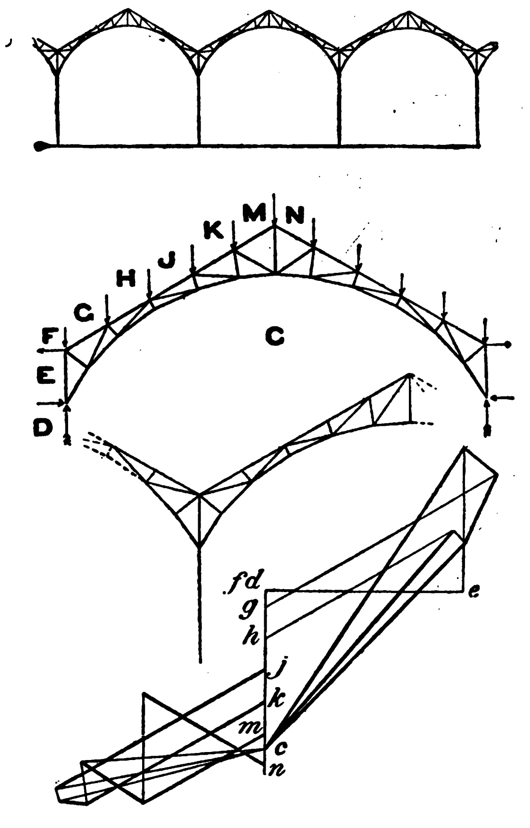


FIG. 83.

arithmetically. Half the weight of the central truss will act at the point of application of H J, and if we denote each load by w, and the horizontal distance apart by x, while the

* See Fig. 83A in the Appendix.

vertical distance between D E and E F may be denoted by h, we obtain, by taking moments about the lowest point of the roof—

$$(3\cdot5\, w \times 2x) + (w \times x) = \mathrm{E\,F} \times h\,;$$

or,
$$\mathrm{E\,F} = \frac{8\,w\,x}{h} = \mathrm{D\,E}.$$

The stress diagram may then be drawn, that for one-half the complete truss being shown in the lower part of fig. 83.

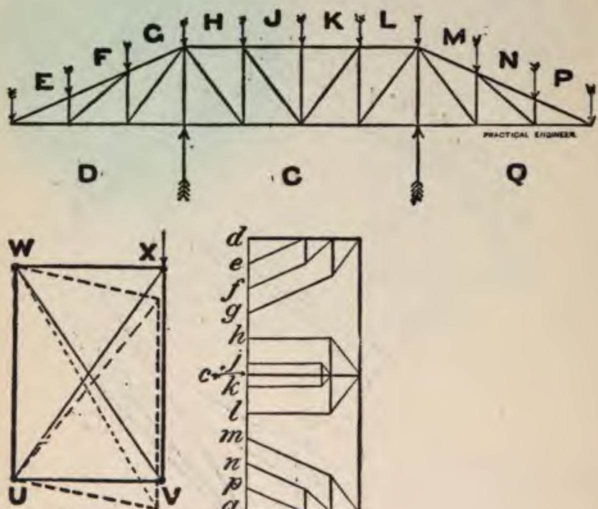


FIG. 84.

It will be noticed that there is no stress in the third member (from the end), which radiates from the centre of the curve of the lower boom; also there is no stress in either of the members attached to its lower end. This portion of the bracing is sometimes replaced by a solid web plate.

On examination of the stress diagram, the first two diagonals on either side of the vertex will be found to be struts; and as it is generally desirous to make the struts as short as possible, or to make the longest members ties, these diagonals should be reversed, together with the

diagonal nearest to the columns, as shown in the third view from the top of fig. 83. The long bracing members are all ties, and the short ones struts. A specimen of this sort of truss may be found in the Liverpool Street Station, London.

CHAPTER XIX.

Double Cantilever Roof Truss.

Another form of truss largely used for covering the platforms of railway stations is that shown in fig. 84. It is supported at two intermediate points, with a considerable portion of the roof overhanging the supporting columns. The truss in the figure has N bracing, and the stress diagram is shown below the truss. The effect of wind on the truss is shown in fig. 85, the direction of the wind being from left to right. The stress diagram for wind pressure alone is given immediately below the diagram of the truss, the one on the left being drawn with the assumption that the right-hand column sustains no horizontal load; and the right-hand diagram, with the assumption that the directions of the reactions of the columns are parallel to one another. The difference is very slight. Referring to the stress diagram, fig. 84, it will be noticed that the verticals are struts, and the diagonals are ties. The wind pressure stress diagram shows that the two diagonals on the right of fig. 85 are in compression, while the verticals are in tension. Should the compressive stress due to wind pressure exceed the stress due to dead load, the diagonals must be designed to resist compression as well as tension. If the diagonals are long, it is customary to insert a second set of diagonals (shown with heavy dotted lines) in the reverse direction, both sets being ties. The reason for this will be more apparent on reference to fig. 84. The figure W X V U represents a single panel with hinged joints, and, for our purpose, fixed at W and U. If a force is applied at X in the direction of the arrow, it will tend to deform the panel into that shown with heavy dotted lines. The diagonal W V will be stretched, and consequently in tension, while the diagonal U X must be in compression, if there is one there

at all. But if this diagonal is designed as a tie, and of a good length, it will bend easily before offering much resistance to compression; and hence we may say that the diagonal W V is the only one that materially tends to resist deformation of the panel. If now the force at X is reversed, the diagonal U X is the one that effectively resists deformation of the panel. As will be seen further on, this double bracing is more general in bridge construction than in that of roofs; but all the same, if by any means the stress in a diagonal is reversed in sign, a reversed diagonal (generally called a counter-brace) will ensure the stability of

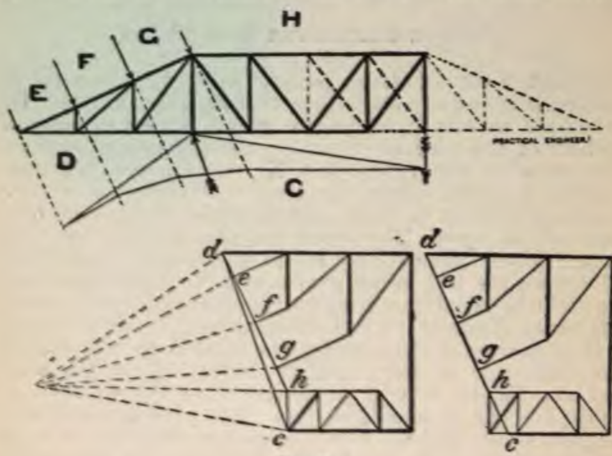


FIG. 85.

the panel. The introduction of counter bracing must not be confounded with the superposition of one truss upon its complementary form; the former is to maintain the stability of one or more particular panels, when the stresses in the diagonals are reversed (generally by means of a movable load), and the latter is a method of producing an increase in strength by a multiplication of parts, rather than by an increase in the dimensions of the individual parts. It is a common occurrence for the crescent-shaped truss to be constructed with double diagonal bracing.

No hard and fast rule can be laid down as to the fitness of any particular truss for a given span, except that the arch,

the quadrangular, or the crescent-shaped truss is invariably used for very large spans, such as terminal railway stations or very large public buildings. The examples that have been given of the different trusses are not of very large dimensions, or for any very special purpose, and hence the designs have been exceedingly simple. The reader is referred to Mr. Bow's work on "Economics of Construction" for the skeleton diagrams of an almost indefinite variety of roofs, and to the pages of *Engineering* and the *Engineer* for the actual designs of roofs of every description.

In the majority of examples of roof details already given, the joints have been arranged with pins, and the axes of

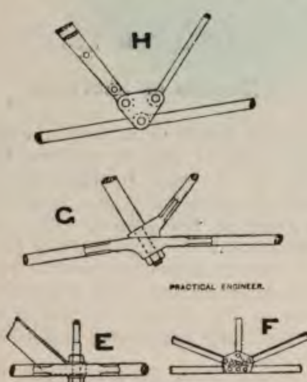


FIG. 86.

the members have all intersected in the same point, at the centre of the pin; but it is a common occurrence in trusses of small span for the joints to be formed other than as already shown, four varieties of which are shown in fig. 86. That depicted at G must be used for small spans only, as induced stresses due to bending would occur in most of the members on screwing up the joint, if the ends are not forged with scrupulous care to their respective shapes. The lowest member is one continuous rod, stretching clear across the roof, and consequently cannot be made conveniently in great lengths. The joint at E is often used in the "king rod" truss, where there are a series of vertical tie rods, and another series of inclined struts.

The joint at F is a common form to be met with in almost all forms of trusses, the several members being held together by means of flitch plates and rivets. A modification of this, in which the rivets are replaced by pins, is that at H; but a better joint would be made by separating the ends of the lowest rods until the centres of all the pins were situated on the circumference of a circle, of which the centre of the present lowest pin is the centre.

CHAPTER XX.

Method of Sections.

As we pass from the truss of a roof to that of a bridge it is usual to call it a girder, and in general there will be two of them, though in railway bridges there may be any number. When each girder has pin joints throughout, and friction be neglected at the joints, while at the same time there are no redundant members—that is, there are only just enough members to produce statical equilibrium—or if any one member be removed, the structure will collapse; then the stresses in the several members can generally be determined by ordinary graphical methods such as those previously described.

But it is often desirable to know the influence of any definite portion of a load in straining any particular member. This may be most easily determined by the application of Ritter's "Method of Sections," and as it also contains the elements of the theory of counter bracing, an example of the method will be now given.

Take a girder made up of symmetrical N bracing, as shown in fig. 87, having six panels, and loaded on the bottom boom or flange. The total load at each joint will be made up of the dead load due to the weight of the girder, flooring, and ballasting, together with the load which is put on, such as a body or number of bodies moving over the bridge. In the present instance we shall only consider the latter element, as the stresses due to the dead load may be added on afterwards, and can nearly always be found graphically. Another and obvious system of nomenclature is also here

introduced for the first time, namely, that of denoting each member by a single letter or number situated in a small circle somewhere near the middle of the member. This is convenient when using the method of sections. In fig. 87 a girder has been chosen with equal panels, the width of each being denoted by b, and the depth by d; also the perpendicular distance apart of two consecutive diagonals has been designated p. Let the several loads be w_1, w_2, &c., not necessarily equal, but situated at the joints; while R and Q are the two supporting forces or reactions. The stress in any member will be denoted by f, with the suffix letter of that member; thus, the stress in a will be denoted by

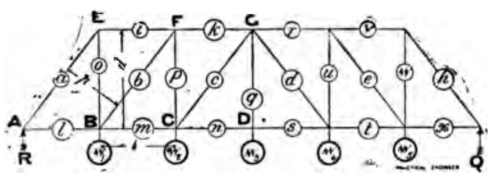


FIG. 87.

f_a, and so on. By taking moments about the abutments, we find that—

$$R = \tfrac{5}{6} w_1 + \tfrac{4}{6} w_2 + \tfrac{3}{6} w_3 + \tfrac{2}{6} w_4 + \tfrac{1}{6} w_5;$$

and $$Q = \tfrac{1}{6} w_1 + \tfrac{2}{6} w_2 + \tfrac{3}{6} w_3 + \tfrac{4}{6} w_4 + \tfrac{5}{6} w_5.$$

These quantities might have been put down from inspection; thus, the weight w_1 is one-sixth of the span from R and five-sixths from Q; hence five-sixths of w_1 will be supported at R, and one-sixth at Q. In the same way, two-sixths of w_2 will be supported at Q, while four-sixths will be supported at R. Now apply the method of sections to obtain the stresses in each individual member.

Cut the first panel on the left with a vertical line, and take moments about E; also denote the sum of these moments by M_E. From the second law of equilibrium, we get—

$$M_E = 0 = f_l\, d - R\, b,$$

and substituting for R from above, we have—

$$f_l = \frac{b}{d} \left(\tfrac{5}{6} w_1 + \tfrac{4}{6} w_2 + \tfrac{3}{6} w_3 + \tfrac{2}{6} w_4 + \tfrac{1}{6} w_5\right).$$

Also $$M_B = 0 = f_a\,p + R\,b,$$

or $$f_a = -\frac{b}{p}\left(\tfrac{5}{6}w_1 + \tfrac{4}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{2}{6}w_4 + \tfrac{1}{6}w_5\right).$$

Now cut l, 0, and i by a straight line; then—

$$M_B = 0 = R\,b + d\,f_i,$$

or $$f_i = -\frac{b}{d}R = -f_l.$$

Proceeding in this way, we shall find that—

$$f_m = f_k, \text{ and } f_r = f_t, \text{ also } f_b = f_x.$$

Again $$M_A = 0 = d\,f_i + b\,f_o,$$

or—

$$f_o = -\frac{d}{b}f_i = -\frac{d}{b} \times \frac{b}{d}\left[-\left(\tfrac{5}{6}w_1 + \tfrac{4}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{2}{6}w_4 + \tfrac{1}{6}w_5\right)\right]$$
$$= R.$$

This might also have been obtained from inspection; thus, the member l cannot transmit any of the vertical force R; therefore it is left entirely to the member a; in fact, the vertical component of f_a must be R. The member i cannot resist any of the vertical component of f_a, and consequently it must be borne by the member 0; or in other words, $f_o = R$.

Further, cutting the second panel with a vertical line, we have—

$$M_F = 0 = 2\,R\,b - w_1 b - d\,f_m,$$

or $$f_m = \frac{b}{d}(2\,R - w_1) = \frac{b}{d}\left(\tfrac{4}{6}w_1 + \tfrac{8}{6}w_2 + \tfrac{6}{6}w_3 + \tfrac{4}{6}w_4 + \tfrac{2}{6}w_5\right),$$

and $$M_A = 0 = w_1 b - f_b\,p + f_i\,d,$$

or $$f_b = -\frac{b}{p}\left(-\tfrac{1}{6}w_1 + \tfrac{4}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{2}{6}w_4 + \tfrac{1}{6}w_5\right).$$

In this way the stress in each member may be found. Some of them are tabulated below. Thus—

$$f_a = -\frac{b}{p}\left(\tfrac{5}{6}w_1 + \tfrac{4}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{2}{6}w_4 + \tfrac{1}{6}w_5\right)$$

$$f_b = -\frac{b}{p}\left(-\tfrac{1}{6}w_1 + \tfrac{4}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{2}{6}w_4 + \tfrac{1}{6}w_5\right)$$

$$f_c = -\frac{b}{p}(-\tfrac{1}{6}w_1 - \tfrac{2}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{2}{6}w_4 + \tfrac{1}{6}w_5)$$

$$f_d = -\frac{b}{p}(\tfrac{1}{6}w_1 + \tfrac{2}{6}w_2 + \tfrac{3}{6}w_3 - \tfrac{2}{6}w_4 - \tfrac{1}{6}w_5)$$

$$f_e = -\frac{b}{p}(\tfrac{1}{6}w_1 + \tfrac{2}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{4}{6}w_4 - \tfrac{1}{6}w_5)$$

$$f_h = -\frac{b}{p}(\tfrac{1}{6}w_1 + \tfrac{2}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{4}{6}w_4 + \tfrac{5}{6}w_5).$$

In the same way, for the verticals, we have—

$$f_o = \tfrac{5}{6}w_1 + \tfrac{4}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{2}{6}w_4 + \tfrac{1}{6}w_5$$

$$f_p = -\tfrac{1}{6}w_1 + \tfrac{4}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{2}{6}w_4 + \tfrac{1}{6}w_5$$

$$f_q = w_3$$

$$f_u = \tfrac{1}{6}w_1 + \tfrac{2}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{4}{6}w_4 - \tfrac{1}{6}w_5$$

$$f_w = \tfrac{1}{6}w_1 + \tfrac{2}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{4}{6}w_4 + \tfrac{5}{6}w_5.$$

And for the lower boom—

$$f_l = \frac{b}{d}(\tfrac{5}{6}w_1 + \tfrac{4}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{2}{6}w_4 + \tfrac{1}{6}w_5)$$

$$f_m = \frac{b}{d}(\tfrac{4}{6}w_1 + \tfrac{8}{6}w_2 + \tfrac{6}{6}w_3 + \tfrac{4}{6}w_4 + \tfrac{2}{6}w_5)$$

$$f_n = \frac{b}{d}(\tfrac{3}{6}w_1 + \tfrac{6}{6}w_2 + \tfrac{9}{6}w_3 + \tfrac{6}{6}w_4 + \tfrac{3}{6}w_5)$$

$$f_s = f_n$$

$$f_t = \frac{b}{d}(\tfrac{2}{6}w_1 + \tfrac{4}{6}w_2 + \tfrac{6}{6}w_3 + \tfrac{8}{6}w_4 + \tfrac{4}{6}w_5)$$

$$f_x = \frac{b}{d}(\tfrac{1}{6}w_1 + \tfrac{2}{6}w_2 + \tfrac{3}{6}w_3 + \tfrac{4}{6}w_4 + \tfrac{5}{6}w_5).$$

And finally the stresses in the upper boom members may be written down from those of the lower boom members as previously shown. An examination of the above results shows that in general the stresses in the booms increase towards the centre, a result previously shown to be true. Also the stress in any member due to any particular load increases as the distance between the load and member

diminishes ; or the nearer a load is to a member the greater will be the stress in it due to that load. And, lastly, the stress in the booms does not change in kind, as all the terms in each expression are of the same sign.

Now, passing on to the diagonals, and taking any one of them at random, say c, the stress in it f_c contains terms some of which are positive and some negative. Now, if f_c is to be a maximum (negative), the positive terms should be as small as possible, and f_c will be greatest when w_1 and w_2 are zero. Referring to the figure 87, and removing for the moment w_1 and w_2, it will be evident that the girder is fully loaded on one side of the member c. Next take the diagonal b. The stress f_b is a maximum (negative) when w_1 is zero or removed from the girder, and therefore when the girder is fully loaded up to that member. It is, then, evident that as a load passes over a bridge—a train, for instance—the maximum stress of one kind occurs in that member when one end of the load has arrived at that member.

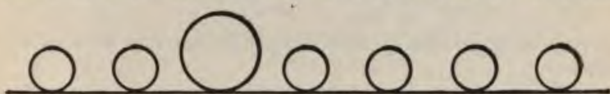
Again returning to the member c, the maximum positive stress occurs when w_3, w_4, and w_5 are zero ; that is, when they are removed from the girder. In the same way the maximum positive stress in b occurs when w_2, w_3, w_4, and w_5 are removed from the girder, and this will happen when the load moves off the bridge in the opposite direction to that previously assumed. It will then be evident that if we consider any diagonal c, and suppose a load approaches the girder from the right, that the negative stress (compressive) increases as the front end of the load approaches c, and is a maximum when the front end is nearest to c, that is at D. The stress will gradually change from negative to positive as the load passes c, and will be a maximum positive (tensile) when the rear end of the load is just leaving c, after which the stress will decrease the further the rear end retreats from c, until it is zero as the load leaves the bridge.

If one load is much in excess of any of the others in the immediate neighbourhood, its influence may so mask that of the others that the above argument appears to fail, and the loads must be shifted on to the next position, and the stress in that member calculated. This can be easily done by merely shifting the weights w_1, w_2., &c., along to their next positions in order, in the expression for f_c, the coefficients remaining in their present places. In this way the influence of the positions of a series of weights can be seen almost at a glance.

The same holds with the vertical members as with the diagonals. The rhythm running through the table of stresses should not be lost sight of, as it enables the remainder of the stresses to be put down after a few have been obtained.

We may sum up the result of the above discussion thus: The maximum, positive, or negative stress in any member occurs when all the individual loads of the opposite sign are removed.

It may be worth while noting that the stress in any one vertical may be written down from inspection. Thus, take the vertical e. The load w_1 is supported five-sixths at R and one-sixth at Q. This latter one-sixth is transmitted from B through the diagonals and verticals to Q, and in so doing it pulls e to the amount $\frac{w_1}{6}$. In the same way w_2 pulls e to the amount $\frac{2}{6}\,w_2$, and w_3 pulls it to the amount



4 4 9 7 5 4 4

FIG. 88.

$\frac{3}{6}\,w_3$, while the $\frac{4}{6}\,w_4$ is also sustained by it. The one-sixth of w_5 that is supported at R tends to compress e; and if we add up all these elements, taking cognisance of signs, we shall have the result previously obtained.

Applying the foregoing investigation to the case of the moving loads shown in fig. 88, the numbers there indicating the loads in tons (and for the sake of simplifying the problem, the distances apart of the loads have been taken the same as the distances between the joints in the girder flange), it is required to find the effect of the position of the load on the stress in the member C, fig. 87, and the position of the load for maximum and minimum stresses. The moving load only is here considered. The loads are assumed to approach the girder from the right. Let the front wheel arrive at D, the middle joint, then w_1 and w_2 are zero, and from the last batch of equations, we find—

$$f_c = -\frac{b}{p}\left(-\tfrac{1}{6}\,w_1 - \tfrac{2}{6}\,w_2 + \tfrac{3}{6}\,w_3 + \tfrac{2}{6}\,w_4 + \tfrac{1}{6}\,w_5\right).$$

o

Inserting the values of $w_1\ w_2$, &c., in this equation, we get—

$$f_c = -\frac{b}{p}(-0 - 0 + \tfrac{3}{6} \times 4 + \tfrac{2}{6} \times 4 + \tfrac{1}{6} \times 9)$$

$$= -4\tfrac{5}{6}\frac{b}{p}\text{ tons.}$$

Now advance the front wheel to C; then—

$$f_c = -\frac{b}{p}(-0 - \tfrac{2}{6} \times 4 + \tfrac{3}{6} \times 4 + \tfrac{2}{6} \times 9 + \tfrac{1}{6} \times 7)$$

$$= -4\tfrac{1}{2}\frac{b}{p}\text{ tons.}$$

When the front wheel is at B,

$$f_c = -5\frac{b}{p}\text{ tons,}$$

and so on, until the fourth load of 7 tons has arrived at A, when—

$$f_c = -\frac{b}{p}(-\tfrac{1}{6} \times 5 - \tfrac{2}{6} \times 4 + \tfrac{3}{6} \times 4)$$

$$= \tfrac{1}{6}\,.\,\frac{b}{p}\text{ tons.}$$

Here the sign has changed from negative to positive, indicating that the stress has been altered from compressive to tensile. In the same way the stresses may be calculated for any and every other position of the load, and the following results may be tabulated :—

When the front load is at W_4, $f_c = -2\dfrac{b}{p}$ tons.

When the front load is at D, $f_c = -4\tfrac{5}{6}\dfrac{b}{p}$ tons.

When the front load is at C, $f_c = -4\tfrac{1}{2}\dfrac{b}{p}$ tons.

When the front load is at B, $f_c = -5\dfrac{b}{p}$ tons.

When the front load is at A, $f_c = -2\tfrac{1}{6}\dfrac{b}{p}$ tons

When the second load is at A, $f_c = -\frac{2}{3}\frac{b}{p}$ tons.

When the third load is at A, $f_c = -\frac{1}{2}\frac{b}{p}$ tons.

When the fourth load is at A, $f_c = \frac{1}{6}\frac{b}{p}$ tons.

When the fifth load is at A, $f_c = 2\frac{b}{p}$ tons.

When the sixth load is at A, $f_c = \frac{2}{3}\frac{b}{p}$ tons.

From this it will be seen that the largest load of 9 tons has the predominating influence in producing stress in the member C; the stress being a maximum as the 9 tons rests on the joint next before it. As that load proceeds from this position across the girder the stress in C diminishes, and passes through zero, at the same time changing sign; then it increases up to a certain quantity, and finally diminishes again to zero as the last load leaves the girder. With the above considered loads, the stress varies from a maximum thrust of $5\frac{b}{p}$ tons, to a maximum tension of $2\frac{b}{p}$ tons.

It must be evident that if there are a number of large loads intermixed with the smaller loads, there will be considerable fluctuation of strsss occurring many times as the loads pass rapidly over the girder, and it is not difficult to imagine cases in which the fluctuations are so extremely rapid as to be almost instantaneous.

As a load may advance towards the bridge in either direction, it is only necessary to calculate the stress in one-half of the members, the other half being similar and symmetrically placed.

With a uniform load the calculations become rather more simple, and the results can be written down from inspection after a few have been obtained.

CHAPTER XXI.

COUNTER-BRACING.*

It is now proposed to show where counter-braces should be introduced; but perhaps it would be as well to first show, without the aid of a calculation, how it is that the stress in a diagonal may change sign as the load passes over the bridge. In the upper part of fig. 89, the load is shown approaching the panel whose diagonal member is E F. That portion of the load on the bridge is sustained by the reactions at the buttresses. The left-hand reaction tends to displace that portion of the girder to the left of E F in the

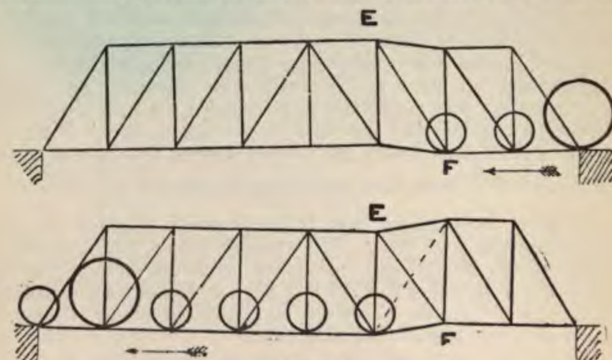


Fig. 89.

upward direction, while the load tends to displace the remainder in a downward direction. This action, which is a shearing action, tends to distort the third panel from the right into the shape shown, in which E F is elongated, and consequently will be in tension. Now let the load proceed across the girder until in the position shown in the lower portion of fig. 89. Here the reverse action has taken place, and E F has been shortened, and consequently must be in compression. Of course, the same action takes place in all positions of the loads, but these two positions are perhaps the simplest and the most self-evident. In the lower part of fig. 89 the contraction of E F may be prevented by

* See Appendix.

inserting the other diagonal, shown dotted, which will be a tie. This latter diagonal is called a *counter-brace*. The effect of the dead load in the above case was not considered.

Whenever the stress in a diagonal changes sign under the influence of *both* the live and dead loads together, it is customary to insert a counter-brace in the same panel; because, in general, it is more convenient and cheaper to use a counter-brace than design the diagonal as a strut, and the weight of the structure is maintained at a minimum.

Take a girder of the form shown in the upper part of fig. 90, whose uniform dead load is assumed at 1·5 tons per panel, and the live load the same as in the previous example, fig. 88. It is required to find how many panels should be counter-braced. Call the diagonals a, b, c, &c., beginning at the left-hand end. The live loads will move along the upper boom, and the dead weight of the structure may be assumed to be concentrated at the joints in the upper boom also. There are ten panels, and, consequently, the denominator of all the fractions in the previous equations* will be 10. This greatly facilitates the calculation, as the decimal notation can be easily applied. We then have—

$$f_a = \frac{b}{p}(\cdot 9\, w_1 + \cdot 8\, w_2 + \cdot 7\, w_3 + \cdot 6\, w_4 + \cdot 5\, w\ \ + \cdot 4\, w_6 + \cdot 3\, w_7 + \cdot 2\, w_8 + \cdot 1\, w_9)$$

$$f_b = \frac{b}{p}(-\cdot 1\, w_1 + \cdot 8\, w_2 + \cdot 7\, w_3 + \cdot 6\, w_4 + \cdot 5\, w_5 + \cdot 4\, w_6 + \cdot 3\, w_7 + \cdot 2\, w_8 + \cdot 1\, w_9)$$

$$f_c = \frac{b}{p}(-\cdot 1\, w_1 - \cdot 2\, w_2 + \cdot 7\, w_3 + \cdot 6\, w_4 + \cdot 5\, w_5 + \cdot 4\, w_6 + \cdot 3\, w_7 + \cdot 2\, w_8 + \cdot 1\, w_9)$$

and so on. In each of these equations, if we consider the dead loads first, $w_1 = w_2 = w_3$, &c., $= 1{\cdot}5$ tons; and the numerical results will be found to be as follow :—

$$f_a = 6{\cdot}75\,\frac{b}{p}\ \text{tons};$$

$$f_b = 5{\cdot}25\,\frac{b}{p}\ \text{tons};$$

$$f_c = 3{\cdot}75\,\frac{b}{p}\ \text{tons};$$

$$f_d = 2{\cdot}25\,\frac{b}{p}\ \text{tons};$$

$$f_e = \cdot 75\,\frac{b}{p}\ \text{tons.}$$

† See Appendix. * See page 190.

Now consider the moving loads. By inspecting the above equations, it will be noticed that the expression which contains the greatest number of negative terms represents the stress in the diagonal nearest the centre of the span, viz., f_e. This diagonal, then, has the greatest chance of having its stress changed in kind from tension to a thrust; and as we move towards the extremities of the girder the chances of changing sign become less, on account of the decrease in the number of negative terms. If, then, the counter-bracing is to be inserted at all, it will be most required in those panels near the centre of the span.

Naming the diagonals from the left abutment to the centre in the same manner as before, namely, a, b, c, d, and e, we must first proceed to find the maximum stress in each of these members as the loads move across the girder. Let the loads move from right to left and face the same way as in fig. 88. In the previous equation, for f_e put in the values of w_1, w_2, &c., assuming the front of the load to be at H, Q, R, and S in turn. The values found for f_e will be respectively—

$$5{\cdot}3\,\frac{b}{p}\text{ tons, } 8{\cdot}2\,\frac{b}{p}\text{ tons, } 7{\cdot}5\,\frac{b}{p}\text{ tons, and } 7{\cdot}2\,\frac{b}{p}\text{ tons.}$$

Now assume the loads to face in the opposite direction, and to move on to the girder from the left towards the right. The stress in the member e, when the leading load is respectively at S, R, Q, and H, will be—

$$-\,2{\cdot}9\frac{b}{p}\text{ tons, } -\,5{\cdot}3\frac{b}{p}\text{ tons, } -\,4{\cdot}2\frac{b}{p}\text{ tons, and } -\,3{\cdot}6\,\frac{b}{p}\text{ tons.}$$

These, it will be noticed, are of a different sign to those obtained with the loads approaching in the opposite direction.

In the same way find the maximum stress in the member d. With the load moving in the original direction from right to left, and facing the same way as in fig. 88, the stress in d, with the leading wheel at R, will be—

$$f_d = \frac{b}{p}({\cdot}1 \times w_1 - {\cdot}2 \times w_2 - {\cdot}3\,w_3 + {\cdot}6\,w_4 + {\cdot}5\,w_5 + {\cdot}4\,w_6$$
$$+ {\cdot}3\,w_7 + {\cdot}2\,w_8 + {\cdot}1\,w_9).$$

Here $w_1 = w_2 = w_3 = 0$, and

$$f = 10{\cdot}7\,\frac{b}{p}\text{ tons}$$

Similarly, when the leading wheel is at S,

$$f_d = 9{\cdot}3\,\frac{b}{p} \text{ tons,}$$

and after moving through one more panel,

$$f_d = 2{\cdot}9\,\frac{b}{p} \text{ tons.}$$

The member c, treated in the same manner, results in $15{\cdot}2\,\frac{b}{p}$ tons as the maximum positive stress, and $-\,1{\cdot}2\,\frac{b}{p}$ as the maximum negative stress.

Tabulating all these, we have:

Member.	Dead-load stress.	Maximum positive live-load stress.	Maximum negative live-load stress.	Maximum total stress.	Minimum total stress.
	Tons.	Tons.	Tons.	Tons.	Tons.
e	$\cdot 75\,\frac{b}{p}$	$8{\cdot}2\,\frac{b}{p}$	$-\,5{\cdot}3\,\frac{b}{p}$	$8{\cdot}95\,\frac{b}{p}$	$-\,4{\cdot}55\,\frac{b}{p}$
d	$2{\cdot}25\,\frac{b}{p}$	$10{\cdot}7\,\frac{b}{p}$	$-\,2{\cdot}9\,\frac{b}{p}$	$12{\cdot}95\,\frac{b}{p}$	$-\,\cdot 6\,\frac{b}{p}$
c	$3{\cdot}75\,\frac{b}{p}$	$15{\cdot}2\,\frac{b}{p}$	$-\,1{\cdot}2\,\frac{b}{p}$	$18{\cdot}95\,\frac{b}{p}$	$+\,3{\cdot}55\,\frac{b}{p}$

The second, third, and fourth columns simply contain the results already found. The fifth column contains the maximum positive stresses due to the dead and live loads together, which are obtained by adding the numbers in column 2 to those in column 3. The last column contains the minimum stresses due to the live and dead loads together, and are obtained by adding those in column 2 to those in column 4. This table shows that at certain times the member e is strained by a tensile force of $8{\cdot}95\,\frac{b}{p}$ tons, and at other times by a compressive force of $4{\cdot}55\,\frac{b}{p}$ tons; or, in other words, it is sometimes a strut and sometimes a tie. The same happens to the member d; but the member c is

never a strut, because the stress in it fluctuates between $15 \cdot 2 \frac{b}{p}$ and $3 \cdot 55 \frac{b}{p}$, both positive values. To prevent d and e becoming struts the counter-braces TR and EQ are inserted. (See the middle part of fig. 90.)

It must be carefully noted that both the live and dead loads must be considered together in determining the panels to be counter-braced. Were there no dead load, then the whole girder would have to be counter-braced; but as the stress due to the dead load alone increases in the diagonals from the centre to the ends, there will be a diagonal somewhere between the centre and one end, in which the dead-load stress is greater than the live-load negative stress, and

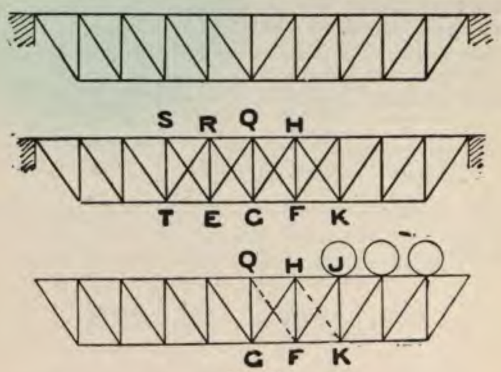


FIG. 90.

consequently this diagonal cannot be negatively stressed—that is, be in compression—under any circumstances of live and dead loads together; and hence no counter-bracing will be required in that or succeeding panels as the end is approached. The counter-bracing being now introduced, it is evident that while each counter-brace tends to prevent the distortion of its panel, the original diagonal will also act in the same way by offering some resistance to compression.

If the diagonal is very long compared with its least transverse dimension, then it will easily bend, and offer very little resistance to compression, and the counter-brace will be the principal factor in resisting distortion of the panel. In general, so that there shall be no tendency to compress

the diagonal at all, the eye at each end is elongated, giving the pin a certain amount of play along the axis of the member. In this way it is ensured that each diagonal can only act in tension.

It is also requisite, after the counter-braces are inserted, to see if they have any influence in changing the stress in either of the remaining diagonals or verticals. Referring to the lowest part of fig. 90, the counter-braces Q F and H K are shown dotted, and a load is shown tending to cause a thrust in G H and the next diagonal on the right. As the negative stress in G H (without counter-bracing) is greater than in its next neighbour, the counter-brace Q F will always be put in if H K is there. The front load will be resisted by the vertical post underneath it, as the diagonal F J cannot act in compression. The thrust in J K will be resisted on the left by a tension in the counter-brace H K; this again will be resisted by a thrust in H F; and further, this will be conveyed to the strut Q G by the counter-brace Q F. To the left of Q G all the diagonals will be in tension, and hence no change in direction of stress will occur.

CHAPTER XXII.

Moment of Inertia.

When discussing the question of the strength of beams and struts, we often had occasion to use the term "Moment of inertia," and it is now proposed to briefly explain its meaning, and how it is obtained. Its value for many different sections has already been given in the table dealing with the strength of struts; and the values for other forms can generally be found in treatises on mechanics or strength of materials; notably Professor Kennedy's "Mechanics of Machinery."

The original idea of the moment of inertia comes to us from a consideration of the dynamics of rotation. Dealing with motion in a straight line we have—

The force acting on a free body = the mass of the free body × the acceleration of the body produced by the force.

Applying this to the particle in fig. 91 (left-hand side), moving in a circle round the centre of motion C, we have—

Force F = mass of particle × its acceleration along the circumference.

If we multiply both sides of this equation by r, the distance of the particle from C, we get—

$$F \cdot r \cdot = M \times \text{linear acceleration} \times r.$$

But the linear acceleration = angular acceleration $\times r$;

$$\therefore \quad F \cdot r \cdot = M \times \text{angular acceleration} \times r^2;$$

or $\quad$ Moment of force = angular acceleration $\times M \cdot r \cdot ^2$

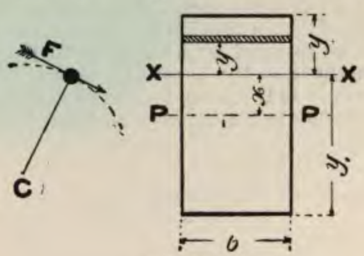


FIG. 91.

The last factor on the right is constant for a certain particle, if its distance r from the centre of rotation is constant, and this factor is called the moment of inertia of the particle.

If we pass from a particle to a composite body, we consider the body to be made up of a very large number of particles, and then the moment of inertia of the body will be the sum of the moments of inertia of all the particles of which the body is made up; or, written symbolically,

$$\text{Moment of inertia} = \Sigma\, m \cdot r \cdot ^2.$$

In the strength of materials we are principally concerned with a section of any particular piece, and there comes into the calculations a factor very similar to the above, viz.,

$$\Sigma\, a \cdot r^2,$$

where a represents a small piece of the area of the section. It will be at once seen that this quantity would be of the

same nature as the one above, if it were multiplied by a thickness and a density; hence the summation in one case is similar to that in the other. The first is called the dynamical moment of inertia, while the other goes by the name of the geometrical moment of inertia. One example only will be given of the accurate method of finding the latter in the case of a rectangular section. That of all other forms of regular figures is exactly similar, but often the expressions are more complicated. In fig. 9 (right-hand side) the moment of inertia of the rectangle is required about the axis X.X., the line P.P. being a parallel line through the centre of gravity of the section. Consider a very thin strip (shown shaded), distance y from the axis X.X., and of width $d\,y$. Its area is $b\,.\,d\,y$., and its geometrical moment of inertia about the axis X.X. = area of strip $\times\, y^2 = b\,.\,d\,y\,.\,y\,.^2$.

Also moment of inertia of whole section about X X

$$= \Sigma\, b\, y^2\, d\, y,$$

taken between limits $-\,y_1$ and y_2

$$= b \int_{-y_1}^{y_2} y^2\,.\,d\,y = \left[\frac{b\,y^3}{3}\right]_{-y_1}^{y_2}$$

$$= \frac{b}{3}\left[y_2{}^3 + y_1{}^3\right] = \frac{b\,(y_2 + y_1)}{3}\left[y_2{}^2 - y_1\,y_2 + y_1{}^2\right]$$

$$= \frac{\text{area of section}}{3}\left[y_2{}^2 - y_1\,y_2 + y_1{}^2\right].$$

If the axis P P coincide with X X, then

$$y_1 = y_2 = \frac{d}{2}$$

where d is the depth of section; and the moment of inertia

$$= \text{area of section} \times \frac{d^2}{12}.$$

This is the moment of inertia of the section about an axis through its centre of gravity, and perpendicular to the dimension d.

If we write x for the distance apart of the lines P P and X X, then the moment of inertia about

$$\text{X X} = \text{area} \times \frac{d^2}{12} + \text{area} \times x^2$$

= moment of inertia about an axis through the centre of gravity + area × square of distance apart of axis X X, and axis through centre of gravity.

This is a property which is often useful for finding the moment of inertia about an axis parallel to another axis through the centre of gravity.

In fig. 92, let the area of the small shaded element be a; then its moment of inertia about K K as axis will be

$$a\,y^2;$$

and, in the same way, its moment of inertia about G G will be

$$a\,x^2.$$

But $$a\,y^2 + a\,x^2 = a\,(x^2 + y^2) = a\,r^2,$$

which is the moment of inertia about an axis through R perpendicular to the plane of the paper. In the right-hand side of fig. 92, the lamina is shown in perspective with the

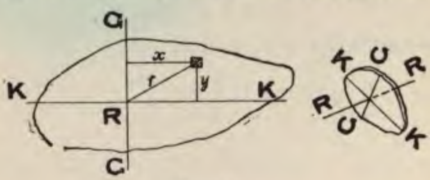


Fig 92.

above axis R R perpendicular to its surface, and piercing it at the point of intersection of the axis K K and G G. As every other element has the same property, and as the moment of inertia of a whole is the sum of the moments of inertia of its parts, we may write the following equation :—

$$I_y + I_x = I_R$$

where the suffixes denote the particular axes about which the moments of inertia are reckoned. This equation enables the moment of inertia of the area of a circle about a diameter as axis to be easily determined, when that about an axis perpendicular to its plane is known.

A very near approximation to the moment of inertia of any plane area may be obtained about a *given axis* by splitting up the area into a number of strips parallel to the above axis, and then adding up the quantities thus—

$$I = a_1\,x_1{}^2 + a_2\,x_2{}^2 + a_3\,x_3{}^2 + a_4\,x_4{}^2 + \&c.,$$

where a denotes the area of one of the strips, and x the distance of its centre of gravity from the above axis. Generally, the moment of inertia is required about the centre of gravity of a section, and in that case the centre of gravity is required to be found. Experimentally, it can be found immediately by simply hanging up the lamina by a piece of cotton or string, the latter being continued beyond the lamina, and holding some weight to keep it taut. Two

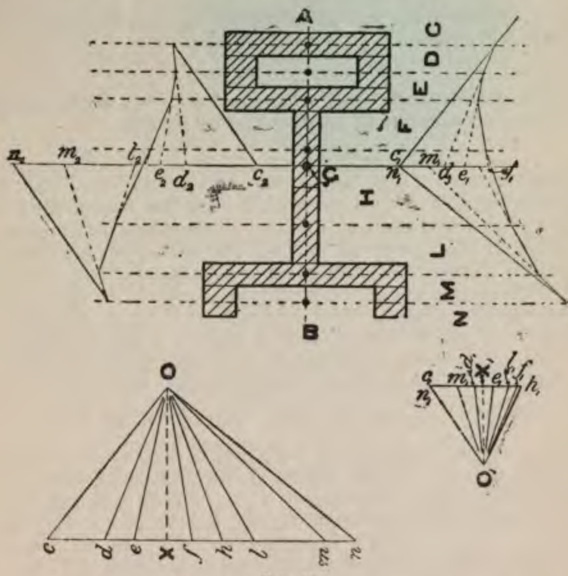


FIG. 93.

points in the lamina covered by the string are noted, and afterwards joined. The same is repeated with some other point of suspension, and the intersection of the two lines thus found gives the centre of gravity of the section. It is generally much quicker to calculate the moment of inertia than to find it graphically; but as some prefer the graphical method, it will now be explained.

*Let it be required to find the moment of inertia of the shaded section, fig. 93, about an axis through its centre of

* See also Appendix.

gravity, which shall be perpendicular to A B. Divide up the shaded area into a number of narrow strips perpendicular to A B. The division lines are shown rather fine. Mark on the centre line A B the centre of gravity of each strip. This is done in the figure by a round dot. Through each of those dots draw a line perpendicular to A B. These are shown dotted. Now, if G is the centre of gravity of the section, and

$$a_1, a_2 \quad . \quad . \quad . \quad . \quad a_7$$

are the areas of the strips respectively, beginning at the A end of A B; also if

$$r_1, r_2 \quad . \quad . \quad . \quad . \quad r_7$$

are the distances of their centres of gravity from the point G, then the moment of inertia of the section

$$= a_1 r_1^2 + a_2 r_2^2 \quad . \quad . \quad . \quad . \quad + a_7 r_7^2$$

approximately. The narrower the strips are, the more accurate will be the final result. In the figure they have been taken very wide, so as not to introduce many lines.

The first operation is to find the centre of gravity G of the cross-section, and this is done by means of the funicular polygon, as explained in a previous article.* Each of the spaces between the dotted horizontal lines is labelled, and the line of loads (areas in this case) $c\,n$ drawn, where $c\,d$ represents the area a_1 to some scale, say y square inches to the inch, the shaded figure being drawn to a scale of, say, $\frac{1}{x}$th full size. Take any pole O (preferably so that O X shall represent some round number), and draw the funicular polygon shown on the right of the shaded section. The line through the intersection $c_1\,n_1$ of the closing lines of the polygon passes through G, the centre of gravity of the section. Now, it has been previously shown that the moment of the load $c\,d$ about the axis through G is given by O H $\times$ $c_1\,d_1$—that is, by O H $\times$ the intercept cut-off on the axis by the two lines in the funicular polygon, which meet in the dotted line through the centre of gravity of the load (strip). In the same way the moment of the second strip about G is O H $\times$ $d_1\,e_1$, and the moment of the third strip O H $\times$ $e_1\,f_1$, and so on for all the strips.

For the sake of rendering the figure less complicated, the intercepts $c_1\,d_1$, $d_1\,e_1$, $e_1\,f_1$, &c., have been repeated in the lower right-hand corner of the figure. This is not necessary

* See page 24.

when the draughtsman is used to the method. Take another pole O_1; the distance $O_1 X_1$, being in the original diagram of lengths, will represent a length to a scale of $\frac{1}{x}$th full size—that is, the real distance $O_1 X_1$ is x times the number of inches in $O_1 X_1$ on the drawing. Complete this second pole diagram, and draw a second funicular polygon (to the left of the shaded section), whose sides are parallel to the radiating lines in the second pole diagram. These sides produced will cut off intercepts $c_2 d_2$, $d_2 e_2$, &c. In the same way as before, the moment of $c_1 d_1$ about the axis through G is $c_2 d_2 \times O_1 X_1$, and the moment of $n_1 m_1$ is $n_2 m_2 \times O_1 X_1$. But the moment of $c_1 d_1$ about axis through G is $c_1 d_1 \times r$; therefore

$$c_2 d_2 \times O_1 X_1 = c_1 d_1 \times r_1.$$

Again, it has been shown above that

$$c_1 d_1 \times O X = c d \times r_1,$$

and, after multiplying these two equations together, we get $c_2 d_2 \times O_1 X_1 \times O X = c d \times r_1^2$ = moment of inertia of upper strip about axis through G. In the same way $d_2 e_2 \times O_1 X_1 \times O X$ is the moment of inertia of the second strip, and as the moment of inertia of the whole equals the sum of the moments of inertia of the parts, $c_2 n_2 \times O_1 X_1 \times O X$ is the moment of inertia of the whole section about axis through G. But in the diagram each part is drawn to a reduced scale, such that the quantity represented by O X is the number of inches in O X on the drawing $\times\ y$ square inches; also the quantity represented by $O_1 X_1$ (being in the diagram of lengths) is the number of inches in $O_1 X_1$ on the drawing $\times\ x$, the result being linear inches. Further, the intercept $c_2 n_2$, being in the diagram of length, will represent the number of inches in $c_2 n_2$ on the drawing $\times\ x$ linear inches; and the moment of inertia of the section will then be given by the number of inches in $c_2 n_2$ on the drawing $\times$ the number of inches in $O_1 X_1$ on the drawing $\times$ the number of inches in O X on the drawing $\times\ x^2 y$.

The moment of inertia may for many dynamical purposes be easily found experimentally as follows: Attach the body rigidly to a length of wire, so that the wire produced would approximately pass through the centre of gravity of the body, and that the wire represents the axis about which the moment of inertia is required. Twist the wire slightly, and note the period of oscillation of the body. Now attach

rigidly to the wire some simple form of body whose moment of inertia is known or can be easily calculated. This is in addition to the body whose moment of inertia is required. Let the moment of inertia of the unknown body be represented by i, and that of the known body by I. Now, the period of oscillation of a body T is connected with its moment of inertia I by the equation—

$$T^2 = c\,I,$$

where c is a constant for a particular wire. Hence, in the case in question,

$$T^2 = c\,I \text{ and } t^2 = c\,(i + I).$$

Dividing one by the other, we get, after simplifying,

$$i = I\left[\frac{t^2 - T^2}{T^2}\right]$$

Radius of Gyration.

Let the moment of inertia of a body be represented by $M\,k^2$; then,

$$M\rho^2 = \Sigma\, m\, r^2\,;$$

and, in the case of an area,

$$A\rho^2 = \Sigma\, a\, r^2 = I\,;$$

then,
$$\rho = \sqrt{\frac{I}{A}}.$$

This quantity ρ is called the radius of gyration. In the case of the rectangle previously alluded to, we have

$$A\,\rho^2 = I = A\,\frac{d^2}{12}\,;$$

therefore,
$$\rho = \frac{d}{\sqrt{12}}$$

when the axis passes through the centre of gravity of the section.

CHAPTER XXIII.

Maximum Bending Moment with Moving Loads.

There are two problems in connection with the maximum bending moment: one to find the maximum bending moment at every point for every position of the moving load; and the other, to find the position of the moving load for maximum bending moment at any point or in any panel of a girder. We will begin by considering the former of these problems.

It has been previously shown in connection with the graphical determination of bending moment,* that the bending moment at any point in a beam for any given position of the loads was given by—the intercept cut-off by the funicular polygon on a vertical line through the point in question × the horizontal pole distance O H × the scale of lengths × the scale of loads. Referring to fig. 94, the bending moment under the load D E is given by $t s$ × O H × scale of lengths × scale of loads.

The funicular polygon for that position of the loads is shown in the figure by the full lines $k l u s y$, while $c g$ is the line of loads; also $c j$ is the reaction at the left abutment, and $g j$ that at the right abutment. Now, whether the loads move along the beam, or the beam moves under the loads in the reverse direction, the relative movement and effect is precisely the same. In any case the funicular polygon will always remain in company with the loads, because its corners are determined by the vertical lines through the loads. The loads and their relative distances apart are quite independent of the length of the beam; hence the load line will always be the same wherever the load is situated; and as the pole O is selected at random, it is independent of the beam, and consequently the pole diagram is entirely independent of the position of the loads along the beam, except the last drawn line O j, which determines the reactions; and therefore the shape of the lower contour of the funicular polygon will always be the same wherever the loads may be situated, but the closing line $k l$ will depend upon the position of the loads along the beam.

Let the loads move along the beam from their present

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* See page 24.

position towards the left. This is the same as keeping the loads stationary, and moving the beam through the same distance to the right. This has been done in the figure for the sake of clearness. The verticals through the ends of the

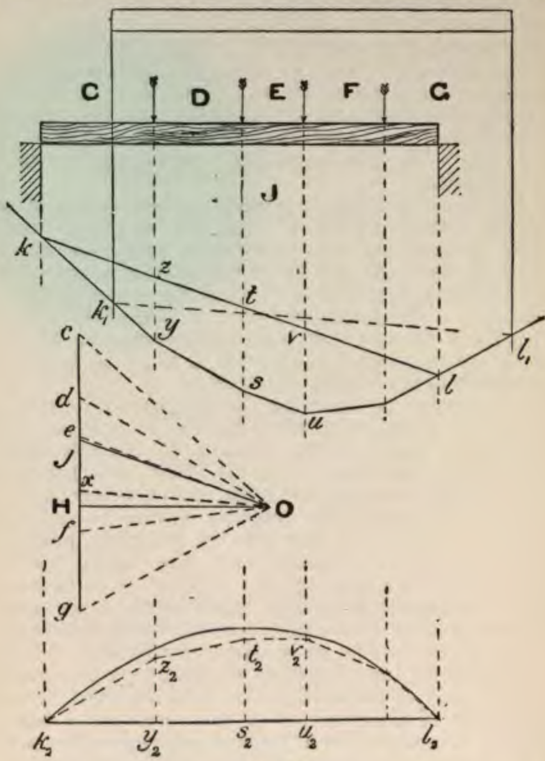

FIG. 94.

beam are also drawn in, cutting the polygon in k_1 and l_1. The closing line is now $k_1 l_1$, shown dotted, and the reactions are $c\,x$ and $g\,x$. The bending moment at any point will be given by the intercept cut-off by the funicular polygon $k_1 y\, s\, u\, l_1 \times$ O H $\times$ scale of lengths $\times$ scale of loads. If now

the beam be shifted into different positions, and a corresponding number of closing lines drawn, the bending moment may be found for all positions of the load. This will be most conveniently accomplished by keeping the beam stationary and shifting the funicular polygon and loads. Under the beam, and parallel with it, draw the line $k_2 l_2$ (lower part of fig. 94), and plot $y_2 z_2$ equal to $y z$, also $t_2 s_2$ equal to $t s$. In that way we get the polygon $k_2 z_2 t_2 v_2 l_2$, whose ordinates are the same as those of the funicular polygon $k y s u l$. Now draw the funicular polygon again for some new position of the loads,

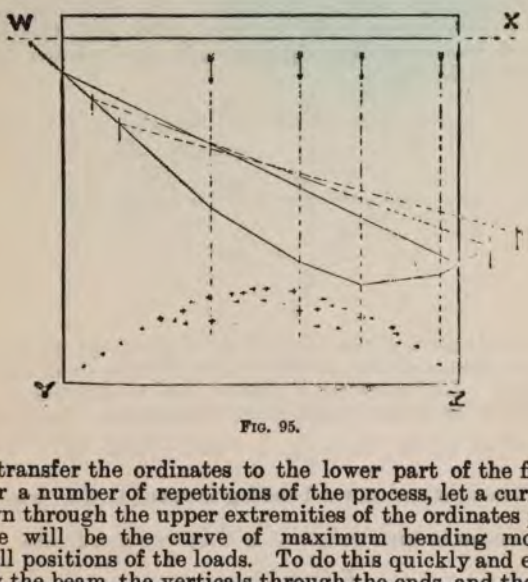

FIG. 95.

and transfer the ordinates to the lower part of the figure. After a number of repetitions of the process, let a curve be drawn through the upper extremities of the ordinates; that curve will be the curve of maximum bending moment for all positions of the loads. To do this quickly and easily, draw the beam, the verticals through the ends, and the base line $k_2 l_2$ on a sheet of drawing paper. These are shown by heavy lines W Y Z X, fig. 95. Then draw the funicular polygon $k y s u l$, fig. 94, on a sheet of tracing paper, together with the vertical dotted lines through the loads and a horizontal line, such as the bottom line of the beam which is also to be found on the drawing paper. This last line is simply for reference, and is shown dotted in fig. 95. The

ends of the polygon should be prolonged for some distance. Now put the tracing paper on the drawing paper, so that the horizontal dotted line on the tracing coincides with the underside of beam, and the end verticals W Y and X Z will cut the funicular polygon. Draw the closing line of the funicular polygon on the tracing, and plot the intercepts on the base line Y X. The polygon is shown in thin lines. Now shift the tracing paper horizontally into a new position. The end verticals will now cut the polygon in two new points, and a second closing line can be drawn, and the ordinates plotted on Y Z, as before. This is repeated for a number of positions, and a number of points are found, as in fig. 95, representing the upper ends of the ordinates. Draw a curve through the extreme points, and this is the curve of maximum bending moment. It is shown with a full line in fig. 94.

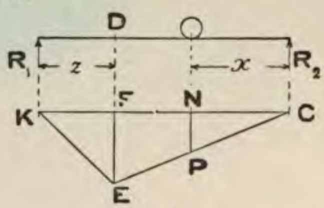

FIG. 96.

We will now find the position of a given set of loads to produce the maximum bending moment at a given point in the span. Let there be a single load w situated at any distance x from the right-hand abutment. The bending moment at D, fig. 96, is $R_1\ z$, which equals $\frac{w\,x\,z}{l}$. If the weight be unity, then the bending moment is $\frac{x}{l}z$, which is the equation to the straight line C E, whose ordinate

$$F\,E = \frac{z}{l}(l - z).$$

In the same way the moment at D, due to the weight unity on the other side of D, will be represented by the ordinate to the line K E. The ordinate N P gives the bending moment at D, caused by the load unity immediately over N; and the moment at D, caused by w in the position shown, is given by w times N P.

Now, if there are a series of loads on the girder, as shown in fig. 97, some of which are on one side of the point D and some on the other side, let G_1 be the sum of the loads on the left of D, and G_2 the sum of those on the right, these resultants acting at the centres of gravity of the weights on each side, and let the figure K E C be the bending moment diagram at D, with unit load at any point along the girder. The bending moment at D due to the load G_2 is

$$G_2 \times y_2,$$

and that due to G_1 is

$$G_1 \times y_1.$$

Therefore total moment at D due to all the loads together

$$= M = G_2 \, y_2 + G_1 \, y_1.$$

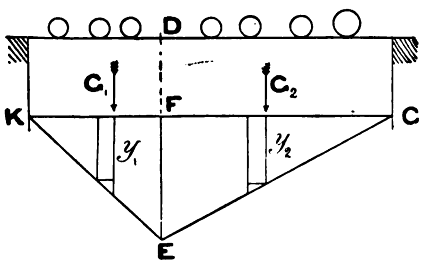

FIG. 97.

Let the loads move towards the left, through a distance $d\,x$; then y_2 becomes

$$y_2 + d\,y_2,$$

and y_1 becomes

$$y_1 - d\,y_1,$$

while M becomes

$$M + d\,M,$$

and

$$M + d\,M = G_2\,(y_2 + d\,y_2) + G_1\,(y_1 - d\,y_1)$$

and, subtracting the previous equation from this, we get

$$d\,M = G_2 \,.\, d\,y_2 - G_1 \,.\, d\,y_1.$$

But $$\frac{d\,y_2}{d\,x} = \frac{\mathrm{F\,E}}{\mathrm{F\,C}} \text{ and } \frac{d\,y_1}{d\,x} = \frac{\mathrm{F\,E}}{\mathrm{F\,K}}$$

$$\therefore d\,\mathrm{M} = \mathrm{G}_2 \cdot \frac{\mathrm{F\,E}}{\mathrm{F\,C}} \cdot d\,x - \mathrm{G}_1 \cdot \frac{\mathrm{F\,E}}{\mathrm{F\,K}} \cdot d\,x$$

or, $$\frac{d\,\mathrm{M}}{d\,x} = \mathrm{F\,E}\left(\frac{\mathrm{G}_2}{\mathrm{F\,C}} - \frac{\mathrm{G}_1}{\mathrm{F\,K}}\right)$$

If M is a maximum, then

$$\frac{d\,\mathrm{M}}{d\,x} = \mathrm{O},$$

and the quantity within the bracket is zero, and therefore

$$\frac{\mathrm{G}_2}{\mathrm{G}_1} = \frac{\mathrm{F\,C}}{\mathrm{F\,K}}$$

and if G denote the total load $\mathrm{G}_1 + \mathrm{G}_2$,

$$\frac{\mathrm{G}}{\mathrm{G}_1} = \frac{\mathrm{G}_1 + \mathrm{G}_2}{\mathrm{G}_1} = \frac{\mathrm{F\,C} + \mathrm{F\,K}}{\mathrm{F\,K}} = \frac{\mathrm{K\,C}}{\mathrm{F\,K}}.$$

These two last equations express symbolically the condition that the point in question divides the loads in the same ratio as it divides the span ; or, if written thus—

$$\frac{\mathrm{G}}{\mathrm{K\,C}} = \frac{\mathrm{G}_1}{\mathrm{F\,K}} = \frac{\mathrm{G}_2}{\mathrm{F\,C}}$$

shows that the average load per foot on either side of the point is the same as the average load per foot on the whole span.

A further problem in connection with the maximum bending moment is to find the position of any one of a given number of loads, so that the bending moment under that load shall be the greatest possible under that load.

In fig. 98 the loads $w_1, w_2 \;.\;.\;.\;.\; w_r \;.\;.\;.\;.\; w_n$ are shown on a girder of total span l. The two reactions are R_1 and R_2, and the distance separating R_1 and w_1 is x, that separating w_1 and w_2 is x_1, and so on, until that between w_n and R_2 is x_n. All the lengths except x and x_n are invariable as the loads move over the bridge. The left reaction is obtained directly by taking moments about the right-hand end, thus—

$$\begin{aligned} \mathrm{R}_1\,l &= w_1\,(x_1 + x_2 + x_3 \;.\;.\;.\;.\; + x_n) \\ &+ w_2\,(x_2 + x_3 + x_4 \;.\;.\;.\;.\; + x_n) \\ &+ w_3\,(x_3 + x_4 + x_5 \;.\;.\;.\; + x_n) \;.\;. \quad + w_n\,x_n, \end{aligned}$$

which may be written

$$R_1 = \frac{1}{l}\Big[x_1 w_1 + x_2 (w_1 + w_2) + x_3 (w_1 + w_2 + w_3)$$

$$\ldots \ldots + x_n (w_1 + w_2 \ldots \ldots + w_n)\Big]$$

The bending moment under the weight w_r is—

$$M_r = R_1 (x + x_1 + x_2 \ldots \ldots + x_{r-1})$$
$$- w_1 (x_1 + x_2 \ldots \ldots + x_{r-1})$$
$$- w_2 (x_2 + x_3 \ldots \ldots + x_{r-1})$$
$$\ldots \ldots - w_{r-1} x_{r-1}$$

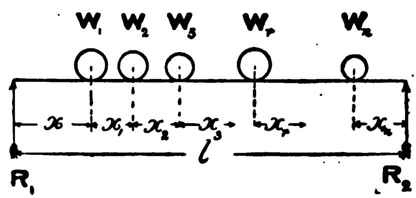

FIG. 98.

which can be written—

$$M_r = R_1 (x + x_1 + x_2 \ldots \ldots + x_{r-1})$$
$$- x_1 w_1 - x_2 (w_1 + w_2) - x_3 (w_1 + w_2 + w_3)$$
$$\ldots \ldots - x_{r-1} (w_1 + w_2 + w_3 \ldots \ldots + w_{r-1}).$$

For a maximum moment under w_r

$$\frac{d\,M_r}{d\,x} = 0.$$

Remembering that in the above equation all the terms are constant except the first, then

$$0 = \frac{d\,M}{d\,x} = R_1 + (x + x_1 + x_2 \ldots \ldots + x_{r-1})\frac{d\,R_1}{d\,x}.$$

The quantity in the brackets is the distance of the load w_r from the left abutment, and which we may call z. Also

the above expression for R_1 contains only one variable namely, x_n, which increases as x decreases; therefore

$$\frac{d\,R_1}{d\,x_n} = \frac{1}{l}(w_1 + w_2 + w_3 \;\ldots\ldots\; + w_n) = -\,\frac{d\,R_1}{d\,x}.$$

Substituting in above, we get

$$R_1 - \frac{z}{l}(w_1 + w_2 + w_3 \;\ldots\ldots\; + w_n) = 0,$$

and

$$z = \frac{R_1\,l}{w_1 + w_2 + w_3 \;\ldots\ldots\; + w_n}.$$

But R_1 contains a term (the last one) which has one factoı a variable, namely x_n. For this substitute its value

$$l - (x + x_1 + x_2 + x_3 \;\ldots\ldots\; + x_{n-1}),$$

and the last equation becomes after replacing

$$z \text{ by } x + x_1 + z_2 \;\ldots\ldots\; + x_{r-1},$$

and

$$w_1 + w_2 + w_3 \;\ldots\ldots\; + w_n \text{ by } w.$$

$$x + x_1 + x_2 \;\ldots\ldots\; + x_{r-1} =$$

$$\frac{1}{w}[x_1\,w_1 + x_2\,(w_1 + w_2) + x_3\,(w_1 + w_2 + w_3) \;\ldots$$
$$+ \{l - (x + x_1 + x_2 \;\ldots\ldots\; x_{n-1})\}$$
$$\{w_1 + w_2 + \;\ldots\ldots\; + w_n\}]$$

or,

$$2\,x = l - [(x_1 + x_2 \;\ldots\ldots\; + x_{n-1})$$
$$+ (x_1 + x_2 \;\ldots\ldots\; + x_{r-1})]$$
$$+ \frac{1}{w}[x_1\,w_1 + x_2\,(w_1 + w_2) + x_3\,(w_1 + w_2 + w_3) \;\ldots\ldots$$
$$+ x_{n-1}\,(w_1 + w_2 \;\ldots\ldots\; + w_{n-1})]$$

After dividing through by 2, add to each side

$$(x_1 + x_2 + x_3 \;\ldots\ldots\; + x_{r-1}),$$

and the above becomes

$$z = \frac{l}{2} - \tfrac{1}{2}(x_1 + x_2 + x_3 \;\ldots\ldots\; + x_{n-1})$$
$$+ \tfrac{1}{2}(x_1 + x_2 + x_3 \;\ldots\ldots\; + x_{r-1})$$
$$+ \frac{x_1 w_1 + x_2(w_1 + w_2) + x_3(w_1 + w_2 + w_3) \ldots x_{n-1}(w_1 + w_2 \ldots w_{n-1})}{2\,(w_1 + w_2 + w_3 \;\ldots\ldots\; + w_n)}$$

The value of the first bracket in this expression is the total length of the loads from end to end, while the value of the second bracket gives the length of the loads from the left-hand end up to the rth individual load under which the bending moment is being considered. The maximum bending moment under the rth load is obtained by substituting for R_1 its value,

$$\frac{z}{l}(w_1 + w_2 + w_3 \quad . \quad . \quad . \quad . \quad + w_n) = \frac{z}{l} w$$

in the previous equation for M_r, which gives maximum—

$$M_r = \frac{w z^2}{l} - x_1 w_1 - x_2 (w_1 + w_2) \quad . \quad . \quad .$$

$$- x_{r-1} (w_1 + w_2 \quad . \quad . \quad . \quad . \quad + w_{r-1})$$

It will be noticed that M_r depends upon z and the number r of the individual load from the left-hand end. Also that in the equation for z all the terms are constant for a given system of loads except the third. This materially facilitates calculation. As an example, let

$$w_1 = 6 \text{ tons}, w_2 = 8 \text{ tons}, w_3 = 7 \text{ tons},$$

$$w_4 = w_5 = w_6 = 5 \text{ tons}.$$

Also let

$$x_1 = x_3 = 8 \text{ ft.}, x_2 = x_4 = 7 \text{ ft.}, \text{ and } x_5 = 6 \text{ ft.};$$

while the total span of the girder is 60 ft. Inserting these values in the above equation for z, and finding the different values of z for maximum bending moment under the first, second, third, and fourth loads respectively, we get—

$$z_1 = 21{\cdot}5 \text{ ft.}, z_2 = 25{\cdot}5 \text{ ft.}, z_3 = 29 \text{ ft.}, \text{ and } z_4 = 32 \text{ ft.}$$

With these different values of z, the corresponding values of M_r are 278 foot-tons, 327 foot-tons, 359 foot-tons, and 306 foot-tons respectively. The greatest of these maxima is that under the third load when that load is 29 ft. from the left abutment. Referring to fig. 94, it will be noticed that the maximum bending moment curve approximates to a parabola, and the error will be very small if a parabola is drawn through the end of the ordinate 359 foot-tons, and through the two ends of the girder.

This method will be found to be the quickest for determining the maximum bending moment that occurs in any

beam or girder, for generally an inspection of the problem will be enough to locate it as under one of two loads or one of three loads. If the number of loads be very great, and irregularly placed, then it may be as easy to determine the maximum bending moment graphically. The second method, namely, that of finding the position of the loads for a maximum moment at any point in the span, may be used to check calculations. Thus, taking the numerical example just given, the position of the loads for a maximum moment at a section 29 ft. from the left abutment is that already found, namely, when the third load is over that section. For in that case

$$\frac{36}{60} = \frac{G}{l} = \frac{G_1}{z} = \frac{G_1}{29} \text{ or } G_1 = 17{\cdot}4.$$

For this to be possible the third load must be over the section in question, for if w_3 be to the right of the section, G_1 will be less than 17·4, and if w_3 be to the left of the section, G_1 will be greater than 17·4.

CHAPTER XXIV.

Design of Plate Girder.

It is required to design a steel plate girder to carry one-half a gantry crane. Span, 40 ft.; maximum load lifted, 30 tons; weight of gantry, 3 tons; maximum deflection to be not more than one five-hundredth of the span, and the girder to be fish-bellied.

It has been previously (page 50) shown that the deflection of a girder can be expressed as

$$\frac{c\,f\,l^2}{d\,E}$$

where f = maximum working stress;
l = length of span;
d = depth of girder;
E = Young's modulus of elasticity,
= 9,000 tons per square inch for built-up wrought iron or steel;
c = a constant depending on the form of girder.

It has been shown that if the girder is of uniform width and uniform strength, loaded at the centre, $c = \frac{1}{3}$; but if the girder is of uniform width, depth, and strength, $c = \frac{1}{4}$.

The case in question does not coincide with either of these cases, but is somewhere about midway between them, so that we may take c about ·3.

It has been shown in connection with working stress that if a load come suddenly on a piece of elastic material, the maximum stress produced is double of that which would be produced if the load were applied steadily, or as a dead load. In the gantry under consideration it is probable that sometimes the lifting chain will slip slightly round the object it is lifting, and thus produce the sudden doubling of the dead-load stress in the chain and girders. Assuming the maximum permissible stress to be 9 tons per square inch over the net section, then the working stress would be 4·5 tons per square inch.

Putting known quantities into the above expression for the deflection, we get

$$\frac{40}{500} = \frac{\cdot 3 \times 4\cdot 5 \times 40 \times 40}{9000\,d}$$

and $$d = 3\text{ ft.}$$

This is for centre of span.

It is customary, in designing a girder, to assume that the bending action is resisted wholly by the flanges, and the shearing action wholly by the web, and in each case the stress is assumed to be uniformly distributed over the section; the flanges to include the angle irons, but no part of the web.

The above assumption may be shown to be approximately true, thus: Take the case we are now dealing with, fig. 99. The area of each flange will be found to be, approximately, 13·5 square inches, and the centre of gravity of each flange is at a point about 17·5 in. from the neutral axis; hence moment of resistance of flanges

$$= 2 \times 17\cdot 5 \times 13\cdot 5 \times 4\cdot 5 = 21400 \text{ tons-inches}$$

The moment of resistance of the web, assuming a thickness of $\frac{3}{8}$ in. at mid span,

$$= \tfrac{1}{6} f\, b\, d^2$$

$$= \tfrac{1}{6} \times 4\cdot 5 \times \tfrac{3}{8} \times 35 \times 35 = 345 \text{ tons-inches.}$$

Hence, in this case, the moment of resistance of the flanges is over sixty times that of the web, and consequently the error is very slight in neglecting that of the web.

With regard to the shearing stress, to show that it may be assumed to be constant throughout any transverse section, consider the stress on the end of the element E A B C D F, fig. 100. Its area $= b\,.\,d\,h$. Total stress over shaded end $= f_2\,.\,b\,.\,d\,h$. from right to left in direction. Total stress over opposite end $= f_1\,.\,b\,.\,d\,h$. from left to right. Resultant of these two end stresses $= (f_2 - f_1)\,b\,.\,d\,h$ in a direction from right to left. Let M_1 and M_2 be the bending moments at distance x_1 and x_2 from the left reaction; then, from previous work [equation (29)],

$$\frac{f}{h} = \frac{M}{I}, \text{ or } \frac{M}{I}h = f;$$

consequently resultant stress on element

$$= \left(\frac{M_2 h}{I} - \frac{M_1 h}{I}\right) b\,.\,dh. = (M_2 - M_1)\frac{b\,.\,h\,.\,dh}{I}$$

from right to left. Also resultant stress on Q M N K T G

$$= \int_{h_3}^{H_2} (M_2 - M_1)\frac{b\,.\,h\,dh}{I} = \int_{h_3}^{H_2} \frac{dM\,.\,b\,.\,h\,.\,dh}{I}$$

when $x_2 - x_1$ is very small, and equal to dx. Stress over area G T K

$$= \frac{\text{resultant force on Q M N K T G}}{\text{area G T K}}$$

$$= \int_{h_3}^{H_2} \frac{dM\,.\,b\,.\,h\,.\,dh}{TK\,.\,dx\,.\,I}.$$

But $\dfrac{dM}{dx}$ = shearing force at distance x from left reaction = F, say, and

$$\int_{h_3}^{H_2} b\,.\,h\,.\,dh.$$

= moment of area M N K T about neutral axis = m, say, = area M N K T × distance of its centre of gravity from the neutral axis.

Then, shear stress over G T K

$$= \frac{F\,m}{b\,I}$$

when b = T K, F = shearing force over whole transverse section, and I = moment of inertia of section. But a shearing stress is always accompanied by another shear stress* of equal intensity, in a plane at right angles to that of the first stress, both planes being perpendicular to the plane of distortion or flexure; consequently in the above beam *at any point* T, the shear stress at a point in a vertical plane is equal to that at the same point in a horizontal plane, both being perpendicular to the plane of flexure.

Thus, in fig. 101, the shear stress at the outer fibre = o, because $m = o$. At B it is

$$\frac{\mathrm{F}}{\mathrm{I}} \times \frac{\text{moment of end of top plate about neutral axis}}{8} = bb_1;$$

* This is easily seen, thus: Take a block of material, shown shaded in the sketch fig. 103. We can produce a shearing stress in it by attaching it to two other pieces of material at its opposite faces, as shown, and then applying two equal and opposite forces P_1 and P_2. By the first law of equilibrium, these two forces together do not tend to translate the shaded material, but they do tend to turn it about an axis perpendicular to the plane of the paper. This couple can be balanced by an equal and opposite couple produced by the forces

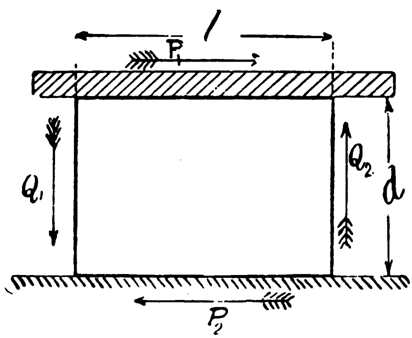

FIG. 103.

Q_1 and Q_2. Let l be the length of the block, t its thickness, and d its depth; then, by the second law of equilibrium,

$$\mathrm{Q}_1 l - \mathrm{P}_1 d = o.$$

Let f_h be the shearing stress in a horizontal plane, and f_v that in a vertical plane, then $\mathrm{P}_1 = \mathrm{P}_2 = f_h \times l \times t$, and $\mathrm{Q}_1 = \mathrm{Q}_2 = f_v \times d \times t$.

Substituting in above for P_1 and Q_1, we find that $f_v = f_h$, and it was shown just above that either of these cannot exist alone; hence, at any point in a piece of material, there must be two shearing stresses perpendicular to each other, and perpendicular to the plane of deformation.

at c it is

$$\frac{\text{F}}{\text{I}} \times \frac{\text{moment of ends of two upper plates about axis}}{8} = c c_1;$$

at d it is

$$\frac{\text{F}}{\text{I}} \times \frac{\text{moment of ends of three plates}}{8} = d d_1.$$

Just below D it is

$$\frac{\text{F}}{\text{I}} \times \frac{\text{moment of three plate ends}}{6\frac{1}{2}} = d d_2;$$

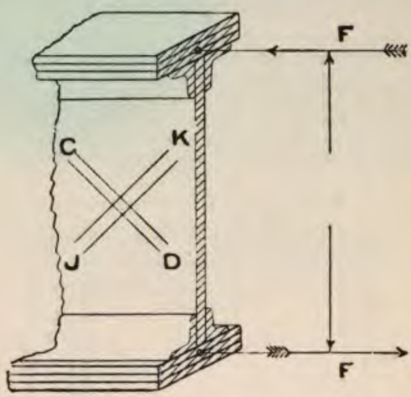

FIG. 99.

just below E it is

$$\frac{\text{F}}{\text{I}} \times \frac{\text{moment of area above E about axis}}{1\frac{3}{8}} = e e_2;$$

just below G it is

$$\frac{\text{F}}{\text{I}} \times \frac{\text{moment of area above G about axis}}{\frac{3}{8}} = g g_2;$$

and at K it is

$$\frac{\text{F}}{\text{I}} \times \frac{\text{moment of area above K about axis}}{\frac{3}{8}} = k k_2.$$

The curved lines are portions of parabolas. This diagram shows at once that the assumption before mentioned is approximately correct, that the web is chiefly instrumental in resisting the shearing force due to the load, and that it is approximately uniformly distributed over the web section.

The effective depth of the girder is taken as the distance between the centres of gravity of the flanges. With this data we have bending moment = moment of resistance of section, or (fig. 99) M = F d where F is the total stress over one flange section. But if A represents the net area of the section of one flange, and f the working stress,

$$f\,\mathrm{A} = \mathrm{F},$$

and $$\frac{\mathrm{M}}{f\,d} = \mathrm{A} = \frac{\mathrm{M}}{4{\cdot}5 \times 36} = \frac{\mathrm{M}}{162} \text{ square inches.}$$

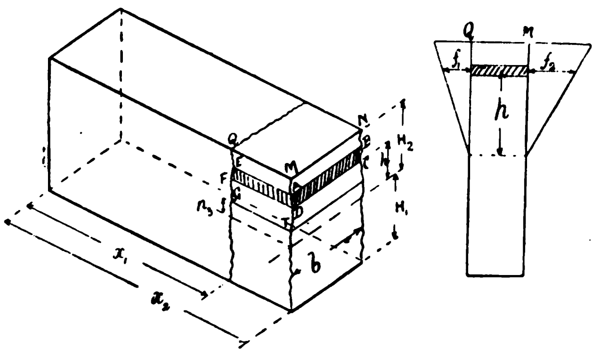

FIG. 100.

Now, if the gantry is situated at a distance x from the left-hand end of girder, the left reaction will be

$$\frac{w\,(l - x)}{l}$$

where w is the concentrated load. The bending moment will be a maximum under the load, and therefore

$$\mathrm{M}_{\max} = w\,(l - x)\,\frac{x}{l},$$

which is the equation to a parabola whose vertex is at mid-span and axis vertical. When

$$x = \frac{l}{2},\ \mathrm{M_{max}} = \frac{w\,l}{4} = \tfrac{1}{2}\,\frac{(30 + 3)\,40}{4} = 165 \text{ tons-feet}$$

$$= 1980 \text{ tons-inches.}$$

But there will be a certain amount of bending moment due to the weight of the girder itself, and this may be determined as follows : Let w be the weight of the girder in tons, $r =$ the ratio of span to depth of girder, and $c = 1{,}200$;

then $$w = \frac{\mathrm{W}\,l\,r}{cf - l\,r}.$$

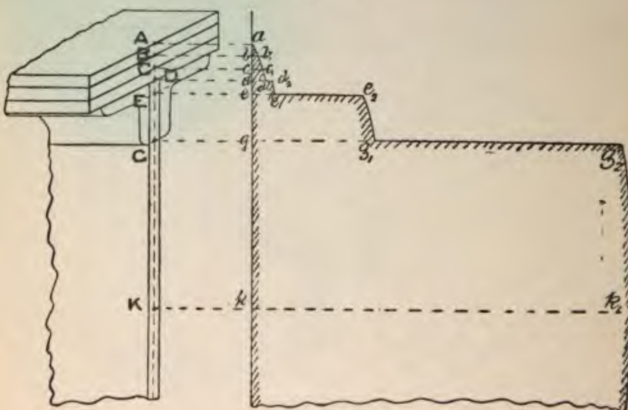

FIG. 101.

Putting in data, we get—

$$w = \frac{\frac{33}{2} \times 40 \times \frac{40}{3}}{1200 \times 4{\cdot}5 - 40 \times \frac{40}{3}} = 3{\cdot}5 \text{ tons.}$$

This weight is, roughly, uniformly distributed over the span, and consequently the bending moment is

$$\frac{w\,l}{8} = 210 \text{ tons-inches.}$$

Adding this to the former moment, we have 2,190 tons-inches as the maximum bending moment at the centre of span. Inserting it in the above equation for A, then

$$A = \frac{2190}{162} = 13{\cdot}4 \text{ square inches.}$$

This will have to be made up of plates and angle irons. The width of the girder is not limited; hence take 8 in. as a convenient width.

Try two $\frac{1}{2}$ in. plates with a pair of $3 \times 3 \times \frac{1}{2}$ in. angles.

Gross sectional area of	two plates	$= 2 \times 8 \times \frac{1}{2} =$	8 sq. in.
" "	two angles	$3 \times 3 \times \frac{1}{2} =$	$5\frac{1}{2}$ "
		Total gross area =	13·5 "

This will clearly be not sufficient when the rivet holes are deducted; hence try three $\frac{3}{8}$ in. plates.

Gross sectional area of	three plates	$= 3 \times 8 \times \frac{3}{8} =$	9 sq. in.
" "	two angles	$3 \times 3 \times \frac{1}{2} =$	5·5 "
		Total gross area =	14·5 "

The diameter of the rivet is given by the equation

$$d = 1{\cdot}2\sqrt{t},$$

where t is the thickness of each plate. With $\frac{3}{8}$ in. plates $d = {\cdot}74$ in., or $\frac{3}{4}$ in. The rivet cuts through three plates and one wing of an angle at the same section, and as there are two rivets, we have, for the area to be deducted on account of rivet holes, $2 \times \frac{3}{4} \times 1\frac{5}{8} = 2\frac{1}{2}$ square inches. After subtraction there are left 12 square inches. This is too little; hence try three $\frac{1}{2}$ inch plates.

Gross area of three $\frac{1}{2}$ in. plates	=	12 square inches.
" of two angles	=	5·5 square inches.
Gross area	=	17·5 square inches.

Using above equation we find that $d = {\cdot}85$, or say, $\frac{7}{8}$ in. Area to be deducted on account of two rivets = 3·5 square inches. After subtraction, we have left 14 square inches. This will do.

In the compression flange the rivets do not appreciably weaken the flange, and consequently they may be left out of account. Two $\frac{1}{2}$ in. plates with two $3 \times 3 \times \frac{1}{2}$ in. angles

will be sufficient, but should there be found a difficulty in arranging bolt holes to attach the rail race for gantry, then $3\frac{1}{2} \times 3\frac{1}{2} \times \frac{1}{2}$ in. angles may be used to advantage. The area of flange would then be $14\frac{1}{4}$ square inches. These will be used if necessary.

With these flange areas we shall have stress intensity × sectional area = F;

$$\text{or stress} = \frac{F}{A}$$

$$= \frac{61}{14} = 4{\cdot}35 \text{ tons for tension flange,}$$

$$\text{and} \quad = \frac{61}{14{\cdot}25} = 4{\cdot}25 \text{ tons for compression flange.}$$

These will be denoted in what follows by f_t and f_c respectively.

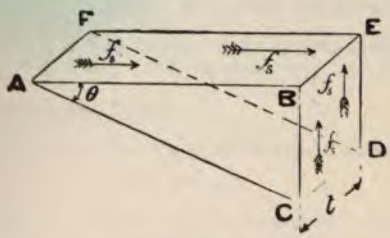

Fig. 102.

Next consider the web thickness at the centre of span. As the shear stress is approximately uniform over a vertical section of the web, we shall have maximum shear force at centre of span = shear stress × area of section.

The maximum shear force at mid span will occur when the load is just on one side of the centre—*i.e.*, when the reactions are approximately equal. Then total load on one girder

$$= \frac{30 + 3}{2} = 16{\cdot}5 \text{ tons.}$$

Shear force at centre = 8·25 tons. There will be no shear force due to the weight of girder, because the weight of is uniformly distributed over the span. Let t be the ss of the web; then $8{\cdot}25 = f_s \times 35 \times t$.

If t is $\frac{3}{8}$ in., f_s = ·63 tons per square inch. If the resistance of the flanges to shearing be also taken into account, the average shear stress f_s will be less than this amount.

It has been mentioned before that at any point in the web there are a pair of shearing stresses of equal intensity, at right angles to one another. These equal shearing stresses produce direct tensile and compressive stresses of equal intensity, inclined to the shear stresses at 45 deg.* We shall thus have strips of the web in compression, such as C D, fig. 99, and consequently have to secure that these strips do not buckle similar to a column. Equation (83), previously given, indicates the limit of direct crushing stress permissible without buckling, it being remembered that all of the strips in question are *fixed* at their extremities, and their lengths = depth of web between angles $\times \sqrt{2}$ Using the values

$$f = 7 \text{ tons}, \; c = \frac{1}{12000}, \; \frac{n\,e}{2} = ·35,$$

$$l = 29 \times \sqrt{2}, \text{ and } t = \rho \sqrt{12},$$

we find that f_c = ·57 ton, which means that if the shear stress in the web at the centre is not allowed to exceed ·57 ton, there will be no chance of buckling in the web. This thickness of web will do at the centre of span. The above value of f allows of a good margin of safety, in fact

---

* Take a triangular prism of material such as A B C D E F, fig. 102, and let there be shearing stresses over the faces A E and C E of intensity f_s. If t be the thickness of prism. then total shear force over C E = B C × t × f_s, and over A E = A B × t × f_s. These shearing forces, which act along planes perpendicular to one another, will produce a stress over the face A D. The direction of this stress in general will be oblique to the face A D, and consequently can be resolved parallel with and perpendicular to that face. As the stress over A D is produced by the stresses over A E and C E, we shall have, after resolving parallel to A C,

$$f_s \,.\, \mathrm{AB}\, t \,.\, \cos\theta - f_s \,.\, \mathrm{BC} \,.\, t \sin\theta = f_{\tan} \,.\, \mathrm{AC} \,.\, t,$$

where $f_{\tan}$ = shearing or tangential stress over A D. Divide through this equation by A C, and we get

$$f_{\tan} = f_s (\cos^2\theta - \sin^2\theta) = f_s \,.\, \cos 2\,\theta.$$

Resolving in the same way perpendicular to A C, we get

$$f_n = f_s \sin 2\,\theta$$

where f_n is the stress intensity perpendicular to A C.

$$\text{Put } \theta = 45 \text{ deg.}: \text{ then } f_{\tan} = 0, \text{ and } f_n = f_s$$

If θ = 135 deg., $f_{\tan}$ = O, and $f_n = -f_s$

The stresses f_n being purely normal to the plane A D, they will be tensile when positive, and compressive when negative, and at right angles to one another.

considerably more than is often allowed. Mr. Cooper, in America, uses the formula—

$$f_c = \frac{5}{1 + \frac{d^2}{3000\,t^2}} \text{ tons per square inch,}$$

where d = depth of web plate. With this formula f_c = $1\frac{1}{4}$ ton per square inch.

As the buckling stress is not reached in the web, there will be no occasion to put in stiffeners to the web near the centre of span. At the extremities let the depth of web plate be 18 in. That part between the angles will be 12 in. deep, and assuming a $\frac{3}{8}$ in. plate, we get as the safe crushing stress 2·07 tons per square inch.

$$\begin{aligned} \text{Also } f_s \times \tfrac{3}{8} \times 11 &= \text{maximum shearing force at end of girder,} \\ &= 16{\cdot}5 + 1{\cdot}75 = 18{\cdot}25 \text{ tons;} \\ \text{therefore } f_s &= 2{\cdot}7 \text{ tons per square inch.} \end{aligned}$$

This is considerably greater than 2 tons; hence try a $\frac{7}{16}$ in. plate instead of the $\frac{3}{8}$ in. plate. Then

$$f_c = 2{\cdot}85 \text{ tons, and } f_s = 2{\cdot}3 \text{ tons.}$$

A $\frac{7}{16}$ in. plate will be adopted without stiffeners, but, of course, a $\frac{3}{8}$ in. plate could be used if stiffeners were introduced. Intermediate sections may also be examined, to see if stiffeners are required in their neighbourhood.

Next, to get the pitch of longitudinal riveting. The rivets prevent the flanges from sliding longitudinally over the web. *Take a small piece of the girder at one end, as in fig. 103, of length equal to the pitch of rivets, p inches. Then for the equilibrium of it we must have

$$\mathrm{F}\,d = \mathrm{R}\,p.$$

R will be greatest when the load is near that end, and will equal 18·25 tons. At the ends d = 19 in., and F will equal the resistance of the rivet to shear, because all the stress in the flange has to be transferred to the web, through the rivet. The rivet will shear at two sections; hence

$$\mathrm{F} = 6 \times 2 \times \frac{\pi}{4} \times (\tfrac{3}{4})^2 = 5{\cdot}3 \text{ tons;}$$

also

$$p = \frac{\mathrm{F}\,d}{\mathrm{R}} = \frac{5{\cdot}3 \times 19}{18{\cdot}25} = 5{\cdot}5 \text{ in.}$$

*For Fig. 103 see Appendix.

See if, with this pitch, the bearing pressure on rivet would be excessive. Let f_b = bearing stress; then

$$f_b \times \tfrac{7}{16} \times \tfrac{3}{4} = 5{\cdot}3,$$

or,
$$f_b = 16{\cdot}7 \text{ tons.}$$

According to the experiments of the Research Committee of the Institution of Mechanical Engineers, the bearing stress may be double of the shear stress of the material. If there should be any preference shown for any definite bearing stress f_b, then

$$p = \frac{\mathrm{F}\,d}{\mathrm{R}} = \frac{f_b \times \frac{7}{16} \times \frac{3}{4} \times 19}{18{\cdot}25} = {\cdot}34\, f_b \text{ in.}$$

If 14 tons bearing stress is allowed, $p = 4{\cdot}8$ in.
If 12 tons bearing stress is allowed, $p = 4$ in.
A pitch of 5 in. will do in this case. To get the length of a cover plate for a joint in the flange, proceed thus:
Net area of one plate $= \frac{1}{2}\,(8 - 1\frac{1}{2}) = 3\frac{1}{4}$ square inches. Maximum total tension in plate $= 3\frac{1}{4} \times 4{\cdot}35 = 14{\cdot}2$ tons. This is resisted by n rivets on each side of the joint, and each rivet is in single shear; therefore

$$14{\cdot}2 = n \times \frac{\pi}{4} \times (\tfrac{3}{4})^2 \times 6,$$

and
$$n = 5{\cdot}3;$$

that is, there must be six rivets on each side of joints. These are situated in two rows, longitudinally; hence the minimum length of cover plate is six pitches, or 30 in.

It will be necessary to know the section beyond which the third plate in the flange is no longer necessary. Let this section be situated y feet on either side of mid-span. At this section net area of flange $= 13{\cdot}4 - 3{\cdot}25 = 10{\cdot}15$ square inches. Total stresses over flange $= 10{\cdot}15 \times 4{\cdot}5 = 46$ tons. The contour of the lower edge of girder is roughly a parabola, in which $y^2 = 4\,a\,x$, when y is horizontal ordinate from centre, and x the vertical ordinate from a tangent at the vertex. To get the value of $4\,a$, take the point $y = 20$ ft., $x = 1\frac{1}{2}$ ft.; then $400 = 4\,a \times 1{\cdot}5$, and $4\,a = 266$; also $y^2 = 266\,x$. But the depth of the girder

$$d = 3 - x$$
$$= 3 - \frac{y^2}{266}\text{ ft.};$$

therefore, moment of resistance

$$= 46\left(3 - \frac{y^2}{266}\right) \text{ tons-feet,}$$

$$= \text{bending moment,}$$

$$= \mathrm{R}\left(\frac{l}{2} - y\right) + \text{bending moment due to weight of girder}$$

$$= \mathrm{R}\left(\frac{l}{2} - y\right)\left(1 + \frac{210}{1980}\right)$$

$$= 1{\cdot}1\,\mathrm{R}\left(\frac{l}{2} - y\right) \text{ tons-feet,}$$

$$\mathrm{R} = \frac{\mathrm{W}}{2l}\left(\frac{l}{2} - y\right)$$

therefore bending moment

$$= \frac{1{\cdot}1\,\mathrm{W}}{2l}\left(\frac{l}{2} - y\right)\left(\frac{l}{2} + y\right)$$

$$= \frac{{\cdot}55 \times 33}{40}\left(\frac{l^2}{4} - y^2\right) = {\cdot}45\,(400 - y^2) \text{ tons-feet.}$$

Then, as moment of resistance = bending moment,

$${\cdot}45\,(400 - y^2) = 46\left(3 - \frac{y^2}{266}\right)$$

and $$12{\cdot}8 \text{ ft.} = y.$$

That is to say, the top plate must extend at least 12 ft. 10 in. on either side of mid-span.

If plates are at hand 26 ft. long, the top plate may be put in without any joint, and thus a cover plate for it is not required; but should so long a plate be not procurable, then it would be convenient to make a joint at mid-span, and continue the upper plate on either side for 3 ft., more or less, the extra piece to act as cover plate to a joint in the second or third plate; while the cover strip to the joint in the first plate may be extended for about 3 ft. on either side, to act as cover strips to the two remaining joints in the second and third plates.

There will be one or two joints in the angles, according to convenience. Net area of one angle = $2\frac{3}{8}$ square inches.

Total stress over angle section = $2\frac{3}{8} \times 4{\cdot}35 = 10{\cdot}4$ tons. Let n = the number of rivets on either side of joint; then

$$n \times \frac{\pi}{4} \times \left(\frac{3}{4}\right)^2 \times 6 = 10{\cdot}4,$$

and

$$n = 3{\cdot}9, \text{ or 4 pitches.}$$

The joints are shown in fig. 104; the parts cross-hatched indicate the lengthening of plates to act as cover strips. There must be some joints in the web. Let F be the shearing force at the section where the joint is required. A cover strip will be put on either side of the web. Let n be the number of rivets on either side of joint; then

$$f_s \times \frac{\pi}{4} \times \left(\frac{3}{4}\right)^2 \times 2 \times n = \mathrm{F};$$

or

$$n = \frac{\mathrm{F}}{f_s \times \frac{\pi}{4} \times \left(\frac{3}{4}\right)^2 \times 2\,n.}$$

After getting n, see if the bearing stress is too great.

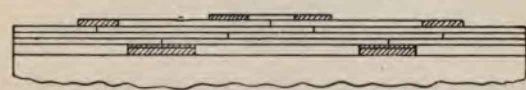

FIG. 104.

Some people prefer to obtain the length of flange plates graphically. In the case of parallel flanges, proceed thus: Set off the span C D, fig. 105, and draw the bending moment diagram, which in the figure is the broken line C E F G H D. The maximum moment occurs under K, and consequently the greatest number of flange plates will occur under K. Let the net area of flange be 20 square inches, of which angles take up 6 square inches, and three equal plates, each of 4·66 square inches. From K set off K N to any convenient scale to represent 6 square inches; also N P = P Q = Q R = 4·66 square inches to same scale. Join R G, and through Q, P, and N draw lines parallel to R G, cutting K G in S, T, and V. Through V, T, S, and G draw lines parallel to C D, and through C, *a*, *b*, *c*, *d*, and D draw vertical lines, thus forming by full lines the outline of the plates. The verticals through C and D must be continued through the

angle iron (*i.e.*, the depth K V) and through the next plate, because the first plate is always continued throughout the length of the girder.

If the girder has other than parallel flanges, the method of procedure is slightly extended. When the flanges are parallel, the bending moment diagram is also the diagram of total flange stress—*i.e.*, a diagram which indicates the quantity F in the discussion on the plate girder. Now,

$$\text{F} \times \text{depth of girder} = \text{moment of resistance,}$$
$$= \text{bending moment,}$$

and
$$\text{F} = \frac{\text{bending moment}}{\text{depth of girder}} = \frac{\text{M}}{d},$$

and it is this quantity F which determines the thickness

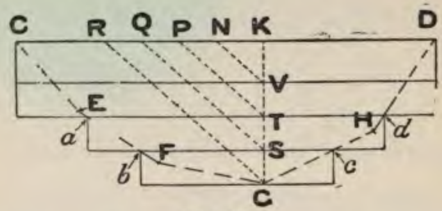

FIG. 105.

and number of plates in a flange, and the length of all the individual plates.

In fig. 106 the depth of the girder varies, that at Q being Q S. The bending moment diagram is the dotted curve passing through s and t, so that the moment at Q S is given by $q\,s$. Now, if the flanges were parallel, the total flange stress F would be represented by $q\,s$, and would equal $q\,s \div$ Q R, or moment ÷ depth. But in the case in question the depth varies and equals Q S, and the total flange stress F equals

$$\frac{\text{moment}}{\text{depth}} = \frac{q\,s}{\text{QS}} = \frac{q\,s}{\text{Q S}} \times \frac{\text{Q R}}{\text{Q R}} = \frac{q\,s}{\text{Q R}} \times \frac{\text{T V}}{\text{Q S}}$$
$$= \text{total flange stress with parallel flanges} \times \frac{\text{T V}}{\text{Q S}},$$

and can be represented by

$$q\,s \times \frac{\text{T V}}{\text{Q S}} = q\,r.$$

In this way the full line curve through r and t is constructed by simply multiplying the ordinate of the bending moment curve by the fraction—

$$\frac{\text{maximum depth of girder}}{\text{depth at section under consideration}}.$$

The full line curve through r and t is then used in the same manner as the curve C E F G H D in fig. 105.

THE "LAUNHARDT-WEYRAUCH" FORMULA.

An expression for the safe working stress, which is often used, especially in America, is that due to Professors Launhardt and Weyrauch.

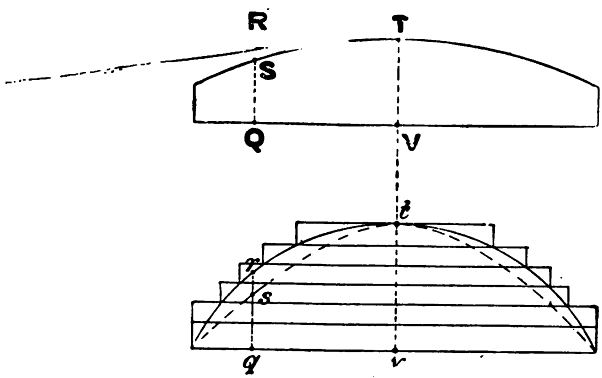

FIG. 106.

Let f denote the working stress, which may be used in designing a structure, when the stress fluctuates between given limits, which may be called maximum and minimum stresses respectively. Also let a represent the constants 5 for structural steel and 4 for wrought iron. Then

$$f = a\left[1 + \frac{\text{minimum stress}}{2\ \text{maximum stress}}\right] \text{tons per square inch.}$$

As an example, take the case of a load varying between 30 tons (tension) and 10 tons (tension). The ratio—

$$\frac{\text{minimum}}{2\ \text{maximum}} = \frac{10}{2 \times 30} = \tfrac{1}{6},$$

and if steel is the material used, we shall then have

$$f = 5\,(1 + \tfrac{1}{6}) = 5\tfrac{5}{6} \text{ tons per square inch.}$$

If the load had varied between 30 tons and 20 tons, then

$$f = 5\left(1 + \frac{20}{2 \times 30}\right) = 6\tfrac{2}{3} \text{ tons per square inch.}$$

Also, if there had been no variation of load, then

$$f = 5\left(1 + \frac{30}{2 \times 30}\right) = 7{\cdot}5 \text{ tons per square inch}$$

If, now, the load fluctuated between 30 tons in tension to 10 tons in compression, we must get

$$f = 5\left(1 - \frac{10}{2 \times 30}\right) = 4\tfrac{1}{6} \text{ tons per square inch.}$$

Again, if the load varies between 30 tons in tension to 20 tons in compression, then

$$f = 5\left(1 - \frac{20}{2 \times 30}\right) = 3\tfrac{1}{3} \text{ tons per square inch ;}$$

but should the load vary between 30 tons in tension to 30 tons in compression, then

$$f = 5\left(1 - \frac{30}{2 \times 30}\right) = 2\tfrac{1}{2} \text{ tons per square inch.}$$

These results agree very well with those observed by Wöhler, previously mentioned.

CHAPTER XXV.

MASONRY STRUCTURES.

THE several members of a masonry structure are connected by mortars of some description or other, whose chief purpose is to completely fill up the space between two adjacent members, so that the pressure between these members shall be uniformly distributed, or shall vary uniformly. Mortars certainly are capable of resisting tension, especially those kinds which go by the name of cement; but it is very small in comparison to their capacity to resist compression. Even if the permissible tensile stress were great, the adhesion of the member to the mortar is small, and consequently the tensile capabilities of the mortar would be of no value, unless some special provision were made by which the mortar could lay hold of the member. To bricks having a coarse surface the mortar clings much more tenaciously than to those having a smooth surface, such as pressed bricks. The following are considered to be the safe working stresses per square inch to which the following material may be subjected, expressed in hundredweights :—

Material	Portland cement.	Brickwork in cement.	Brickwork in mortar.	Concrete.	Common mortar.	Granite.	Limestone.	Sandstone.
Tension	1	·5	·1	·3	—	—	—	—
Compression..	9	1	·5	2	·5	10	9	5

In the preceding table the concrete is assumed to be of the composition of 5 to 1, while the Portland cement in the first column is supposed to be neat. Mortar made with hydraulic lime will withstand a tension of ·1 cwt. per square inch.

With ordinary materials at hand it is customary to design structures which are important, or whose collapse may cause injury to life or property, with the assumption that no part may, under any circumstances, be strained in tension. But in the case of enclosure walls and other similar

structures tension is often allowed, and if of any considerable amount, hydraulic lime mortar or cement is used at the weakest sections.

The conditions, that a structure may be stable, are: That it must not overturn about any joint; the intensity of the working stress in tension and compression must be within the limits of stress prescribed as safe; and that there must be no sliding of one block over another at any of the joints.

Let us take the general case, and refer to fig. 107, where two blocks are in contact, their mutual surfaces having the line C D as a trace in the plane of the paper. Assume tension possible, and that the surfaces are in contact throughout by the aid of cement or other materials. Let the resultant force with which the upper acts on the lower block be represented by R, having its point of application at a distance y from C and x from the centre of gravity of the section G. Resolve R into components H and N, parallel and perpendicular to C D; then H tends to make the upper block slide over the lower one, while the component N produces a variable normal stress over the surfaces in contact. At G, the centre of gravity, introduce two equal and opposite forces N_1 and N_2, both equal to N. Equilibrium is still maintained. N and N_2 together produce a couple whose moment is given by $N x$, while N_1 produces a uniform stress over the section equal to $N_1 \div$ the area of section. But from previous work we have found that

$$N x = \frac{f}{h} I \text{ [see eq. (29)]},$$

because the stress is assumed to vary uniformly; and where f is the stress produced by the couple at a distance h from the centre of gravity of the section, and I is the geometric moment of inertia of the section, about an axis in the plane of section through G and perpendicular to the plane of the paper. The stress f, being plotted on the base line $E_1 E_2$ for different values of h, gives the diagram $E_1 F_1 G_1 F_2 E_2$, where ordinates below $E_1 E_2$ represent compression, and above, tension. Also draw $K_1 K_2$ parallel to $E_1 E_2$ at a distance from it representing, to the same scale, the direct compressive stress $N_1 \div$ area of section. Then at the point E_1 we have the resultant compressive stress $f_c = f_1 + f_2$, and at E_2 the resultant tensile stress $f_t = f_1 - f_2$. If no tensile stress is allowed, then

$$f_1 - f_2 = o, \text{ and } f_1 = f_2.$$

Substituting from the above equation for f_1, and $\frac{N}{A}$ for f_2, we have

$$\frac{N\,x\,h}{I} = \frac{N}{A} \text{ or } x = \frac{I}{h\,A}$$

$$\text{and } y = CG - x = \frac{CD}{2} - \frac{I}{h\,A}.$$

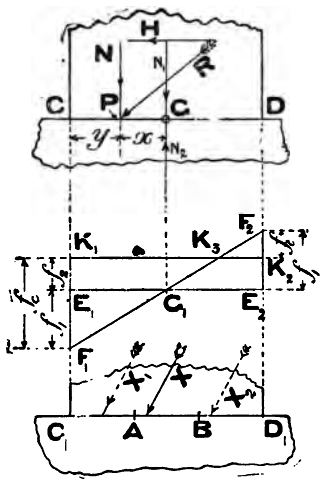

FIG. 107.

If the value of I be inserted from a previous table, it will be found that for a

* Rectangular section .... $y = \frac{1}{3}$ C D ;
Circular section .... $y = \frac{3}{8}$ C D ;

This means, that in the case of a rectangular section, for example, the resultant force R exerted by one block upon its neighbour must not cut the section at a point nearer the outside edge than one-third of the depth of the section, if there is to be no tension, and if the stress varies uniformly over the whole section. As this may happen with regard to

* See Appendix.

either edge, there remains a space, in this case equal to one-third of the depth, inside of which the resultant must pass, consistent with stability under the above conditions. This is generally defined by saying the resultant must pass through the *middle third* of the section if it is rectangular, the *middle fourth* if circular. In the lower portion of fig. 107 A B is the middle third of C_1 D_1. The resultant, as shown at X, indicates a stable structure, while that at X_1 or X_2 indicates instability.

As an example bearing upon the foregoing discussion, take the wall, fig. 108, whose height is 4 ft., made of ordinary

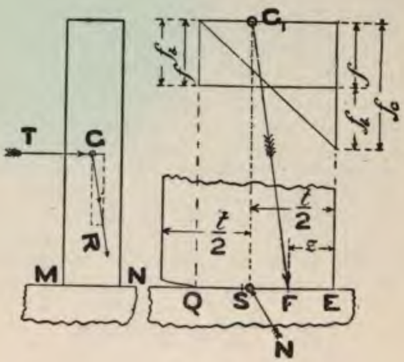

FIG. 108.

stock bricks, and with good mortar, which will safely withstand a compressive stress of 50 lb. per square inch, but will not resist tension. It is required to find what must be its thickness so as to be stable under a wind pressure of 32 lb. per square foot. The weight of 1 cubic foot of the wall is 120 lb.

It has been previously assumed that the *stress varies uniformly over the whole of any horizontal section of the wall* such as M N; consequently the stress at M will be zero, and a maximum at N, and the resultant of the wind pressure T, and the corresponding weight of wall W will be R, and will pass through a point in M N at a distance of $\frac{M\ N}{3}$ from N.

Taking moments about the point of intersection of R and M N, we have—

$$T \times \frac{h}{2} = W \frac{t}{6};$$

where T is the total wind pressure in pounds over a piece of wall 1 ft. long, h inches high, and t inches thick, and which weighs W lb., its centre of gravity being at G.

But

$$W = w \times 1 \times \frac{h}{12} \times \frac{t}{12} \text{ lb.},$$

w being weight of 1 cubic foot; and

$$T = P \times \frac{h}{12} \times 1 \text{ lb.},$$

where P = wind pressure per square foot; therefore

$$\frac{P\,h}{12} \times \frac{h}{2} = \frac{w\,h\,t}{144} \times \frac{t}{6}$$

and

$$\frac{36\,P\,h}{w} = t^2$$

or,

$$t = 6\sqrt{\frac{P\,h}{w}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$

Substituting in this equation from the given data, we get $t = 21{\cdot}4$ in.

We also have the relation: Moment of wind pressure about the middle point of M N = moment of resistance of that section about same point; or,

$$T \, . \, \frac{h}{2} = \tfrac{1}{6} \, . \, f_b \, . \, b \, . \, t^2;$$

where f_b is the stress due to bending, and b the width of piece of wall in question = 12 in. But if there is no tension at M, then at N, f_b will equal

$$\frac{W}{b\,t} = f;$$

and as

$$f_b + f = f_c,$$

the total crushing stress at N, then

$$f_c = 2\,f_b.$$

Substituting in the above equation, we get—

$$f_c = \frac{w\,h}{864} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(B)}$$

and putting in the known quantities,

$$f_c = \frac{120 \times 48}{864} = 6{\cdot}6 \text{ lb. per square inch.}$$

This stress is far below the maximum permissible of 50 lb. per square inch, and consequently the material is not being used to the greatest advantage. At present we have no stress at M due to a wind pressure of 32 lb. per square foot. Suppose the wind pressure to increase, then, as there can be no tension at M, fig. 108, the material there must give way, and the joint will begin to open, as shown on the right-hand side of the figure, and bearing will take place along a portion only of M N, such as Q E, the stress at Q being zero, and that at E being f_c. In this case the stress will vary uniformly over Q E; and as the point Q moves towards E, the point F, where the resultant cuts the joint, will also move towards E, because under the given conditions R must pass through F, and F E = one-third of Q E. The shifting of the point F towards E increases the stability of the wall, because N F is thereby increased, and consequently the moment of the weight of the wall W × N F is also increased. Therefore we may say that the opening of the joint increases the stability of the wall up to a certain limit, and this limit will be seen immediately. As Q moves towards E, the stress at E due to bending increases, and therefore the total crushing stress f_c increases. This may go on until f_c reaches the limit—50 lb. per square inch—in this particular case; in others, the limiting crushing stress; after which, if Q still moves on towards E, the stress at E becoming greater than the limit, the material there will begin to crush, and the joint will give way altogether. This reasoning will account for the stability of masonry structures containing cracks of considerable magnitude. In the figure the opening of the joint is much exaggerated; it will in most cases remain undetected. The problem which now presents itself is to find the thickness of the wall consistent with stability when the windward side of the joint is permitted to open until the stress on the leeward side reaches the maximum allowed. The centre of gravity is shown at G_1, and we shall have moment of wind pressure about F − moment of W about F = O; or,

$$\frac{T\,h}{2} = W \times N\,F = W\left(\frac{t}{2} - Z\right) \quad . \quad . \quad . \quad . \quad (C)$$

The total moment of resistance about F is zero. Also by taking moments about the middle point of Q E, we have

moment of wind pressure − moment of weight of wall − moment of resistance due to bending only = O ; or,

$$\frac{\mathrm{T}\,h}{2} - \frac{\mathrm{W}}{2}(t - d) = \frac{1}{6} f_b \, . \, b \, . \, d^2 \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{D})$$

where f_b is the stress due to bending, and equals $\frac{1}{2} f_c$. From the figure, $3\,\mathrm{Z} = d = \mathrm{Q\,E}$.

Substituting in equations C and D, we get the relations

$$\left.\begin{aligned}
t^2 &= \frac{12\,\mathrm{P}\,h}{w\left[1 - \dfrac{w\,.\,h}{1296\,f_c}\right]} \\
\mathrm{P} &= \frac{t^2\,w}{12\,h}\left(1 - \frac{w\,h}{1296\,f_c}\right) \\
h &= \frac{t^2\,w}{12\ \mathrm{P} + \dfrac{t^2\,w}{1296\,f_c}} \\
f_c &= \frac{w\,h}{1296\left[1 - \dfrac{12\,\mathrm{P}\,h}{t^2\,w}\right]}
\end{aligned}\right\} \quad . \quad . \quad . \quad (\mathrm{E})$$

Taking the numbers just previously used, we find that $t = 13$ in. Thus, while with the assumption that the resultant must pass through the middle third, the least thickness consistent with stability is 21·4 in., and the maximum crushing stress $f_c = 6{\cdot}6$ lb. per square inch. The wall would not collapse if made only 13 in. thick, when the maximum stress would be 50 lb. per square inch.

It will also be found that

$$\mathrm{Z} = \frac{w\,.\,h\,.\,t}{2592\,f_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{F})$$

With the numbers before used, $z = {\cdot}57$ in. Equation (F) may be easily obtained thus :—

The vertical load which produces direct stress f over the area of contact at joint is W. The area of contact is $3\,\mathrm{Z} \times b$, and in general b may be considered as 12 in. for the sake of convenience.

Hence $$f \times 3\,\mathrm{Z} \times b = \mathrm{W}\,;$$

but $$f = \frac{f_c}{2} \text{ and } \mathrm{W} = w \times \frac{h}{12} \times \frac{t}{12} \times 1.$$

Substituting, we get

$$Z = \frac{w\,h\,t}{2592\,f_c} \text{ or } f_c = \frac{\text{vertical load}}{3\,Z \times b \times 72.} \text{ lbs. per sq. in.} \quad . \quad (G)$$

and replacing t by its equivalent in equation (E), we get

$$Z = \frac{h}{2592\,f_c} \sqrt{\frac{12\,w\,P\,h}{1 - \frac{w\,h}{1296\,f_c}}} \quad . \quad . \quad . \quad . \quad . \quad (H)$$

In general it is more convenient to proceed in a slightly different manner to find the thickness of a wall or buttress, though the result is practically the same.

Let it be required to design a pillar 7 ft. high and 18 in. wide, to withstand a thrust of 800 lb. inclined at an angle

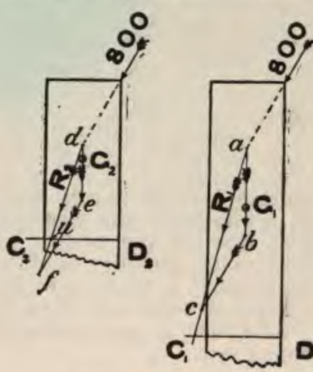

FIG. 109.

of 60 deg. to the horizon, the material used being common brick and mortar, in which the safe stress is 60 lb. per square inch in compression, and zero in tension, and the weight of one cubic foot about 120 lb. On account of the height of the pillar it will be most economical to increase the thickness near the base, and at other sections as required, as shown in fig. 111. Let the thickness at the top of the pillar be 14 in. It is first required to find how far down this thickness may be continued consistent with stability. In fig. 109 let C_1 D_1 be the section below which the thickness must be increased, the distance of C_1 D_1 from the top of the pillar being 4 ft.

Weight of 4 ft. of pillar (when w = weight of 1 cubic foot)

$$= 4 \times \frac{18}{12} \times \frac{14}{12} \times w = 840 \text{ lb.}$$

This will act vertically downwards through the centre of gravity of the mass at G_1. Find the resultant R_1 of the 800 lb. thrust, and the weight of 840 lb. This may be done thus: Produce the line of action of the 840 lb. backwards until it intersects the line of action of the 800 lb. in a. Set off $a\,b$ to represent 840 lb., and through b draw $b\,c$ parallel to, and to represent to the same scale, the thrust of 800 lb. Join $a\,c$, then $a\,c$ represents the resultant R_1 of these two forces to the same scale. The line of action of R_1 cuts $C_1\,D_1$ altogether outside the base; consequently the pillar is unstable under all considerations, and it must be thickened higher up than $C_1\,D_1$. Try the section $C_2\,D_2$ at a distance of 2 ft. 6 in. from the top, fig. 109. The weight of the piece of pillar above $C_2\,D_2$

$$= 123 \times \frac{30}{12} \times \frac{18}{12} \times \frac{14}{12} = 525 \text{ lb.},$$

and the resultant R_2 of this force and the 800 lb. is found to be $d\,f$, which intersects $C_2\,D_2$ in u. The distance $C_2\,u$ = 1·25 in. by measurement = Z in the discussion of the previous problem, equation (G) or (H). We next require to find whether the maximum stress f_c at C_2 is greater than 60 lb. per square inch. In the previous problem it was shown that when no tension is permitted the maximum stress f_c is double the mean stress—*i.e.*, double the uniform compressive stress over the bearing area, due to the vertical component of the load acting through the centre of gravity of the area of contact. That area is 18 in. long (the width of pillar) and 3 Z broad = $3\,C_2\,u$; because the resultant cuts the base such that $C_2\,u$ is one-third of the depth of contact, or one-third of the depth over which there is any stress. We then get

$$\text{mean stress over area of contact} = \frac{\text{vertical component } R_2}{\text{area of contact}}$$

$$\text{or } f = \frac{\text{vertical component of } R_2}{3\,Z \times b.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (K)$$

where b is the width of pillar. But as $f_c = 2f$, then

$$f_c = \frac{2}{3\,Z\,b} \times \text{vertical component of R}$$

$$= \frac{2 \times 1240}{3 \times 1{\cdot}25 \times 18} = 36{\cdot}5 \text{ lb. per square inch.}$$

This is considerably below the limit, and will be adopted. Of course, some other section might be tried, say at a depth of 3 ft. from the top, if desired. After C_2 D_2 let the pillar be thickened by one brick, say $4\frac{1}{2}$ in., as shown in fig. 110, the section C_3 D_3 being the same as C_2 D_2 in fig. 109. The resultant R_2 of the weight of the top block and the thrust is shown. Try the section E_3 F_3 at the bottom of the pillar, and see if it can remain without further thickening. The weight of the lower block is

$$120 \times \frac{54}{12} \times \frac{18{\cdot}5}{12} \times \frac{18}{12} = 1660 \text{ lb.},$$

and this acts vertically through the centre of gravity G_3. Produce the line of action backwards until it intersects

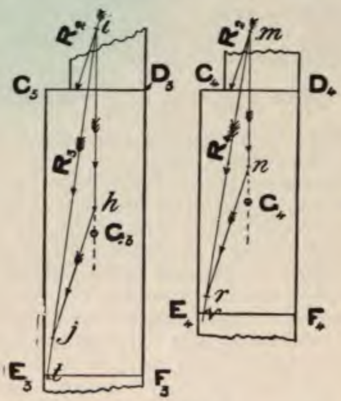

FIG. 110.

R_2 in i. Set off $i\,h$ as before, equal to 1,660 lb., and $h\,j$ parallel and equal to R_2; then $i\,j$ represents their resultant, and cuts E_3 F_3 in t. $E_3\,t = z = \frac{1}{2}$ in., and

$$f_c = \frac{2 \times \text{vertical component of } R_3}{3 \times {\cdot}5 \times 18} = 214 \text{ lb. per sq. inch.}$$

This is too great; hence try the section E_4 F_4, fig. 110, at a distance of 3 ft. 6 in. from C_4 D_4. The weight of material in C_4 F_4 is found to be 1,290 lb., and the vertical component of R_4 is 2,500 lb., and $z = 1\frac{1}{4}$ in., while $f_c = 74$ lb. per square inch. This also is too much; hence try a depth of 3 ft. from

C_5 D_5, fig. 111—that is, at the section E_5 F_5. The weight of the material C_5 F_5 is found to be 1,104 lb., and the vertical component of the resultant R_5 is 2,320 lb., while

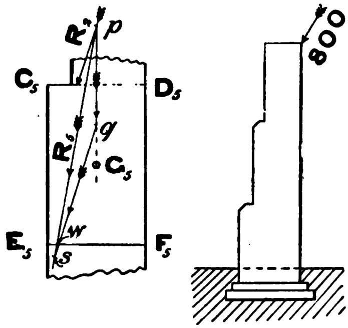

FIG. 111.

Z measures 1·5 in. Inserting these values in the above equation, we find that

$$f_c = 57 \text{ lb. per square inch.}$$

This will do, and the completed pillar is shown in fig. 111.

It will often be found that the process of trial and error, under special conditions, will determine dimensions with quite enough accuracy for practical purposes, and give results without the use of long and tedious formulæ.

STABILITY OF CHIMNEY STACK.

A round brick chimney, shown in section in fig. 112, is built of good brick and to the dimensions given in feet in the table on next page. It is required to find what wind pressure the chimney will withstand, when the stress on the windward side of a section is assumed to be zero, and consequently the resultant will pass through a point at a distance of one fourth of a diameter from the centre of section. The weight of 1 cubic foot of the material of which the chimney is constructed is 100 lb.

The height of the chimney is 125 ft. above M N, and is made in three sections, of lengths shown in the figure. The thickness of brickwork is constant for each of the three pieces into which the chimney is divided. The average

ratio of inside to outside diameter throughout the length of the chimney is about 7 to 10.

Let P be the wind pressure per square foot on a plane surface situated perpendicular to the direction of the wind, then it is found by experiment that the effective pressure on a cylindrical surface is one-half of the pressure P. The point of intersection of the resultant and a horizontal section is called the centre of pressure, and it is the point about which the total moment of resistance of a joint is

	Outside diameter.	Least inside diameter.	Greatest inside diameter.
E F	8·25	6	..
H K	6·5	6·5	7·25
C D	11	7·25	8
M N	12·25	7·25	8·5

zero. This point is at a distance Z from the leeward side of the chimney, and from the previous table of different values of Z for various sections, we find in this case $Z = \frac{1}{4} d$ where d is the outside diameter of the chimney.

First investigate the stability of the top piece at the section H K.

Moment of wind pressure about centre of pressure in H K = average effective pressure per square foot over an axial section $\times \dfrac{40}{2}$ pounds-feet.

$$= \frac{P}{2}\left(\frac{8{\cdot}25 + 9{\cdot}5}{2}\right) \times 40 \times 20 = 3550 \text{ P pounds-feet.}$$

Moment of weight of top piece of chimney about the centre of pressure = weight of top piece $\times \dfrac{d}{4}$.

$$= \frac{\pi}{4}\left[\frac{(8{\cdot}25^2 - 6^2) + (9{\cdot}5^2 - 7{\cdot}25^2)}{2}\right]$$

$$\times 40 \times 100 \times \frac{d}{4} \text{ pounds-feet.}$$

$$= 110{,}500 \times \frac{d}{4} = 262{,}000 \text{ pounds-feet approximately.}$$

As the total moment of resistance of the section is zero about the centre of pressure, we shall have for the equilibrium of the top piece of chimney—

Moment of wind pressure = moment of weight of top piece about centre of pressure,

or $$3550 \text{ P} = 262000$$
and $$\text{P} = 74 \text{ lb. per square foot}$$

As the pressure of the wind is not assumed to reach this magnitude, the chimney is safe against overturning at the

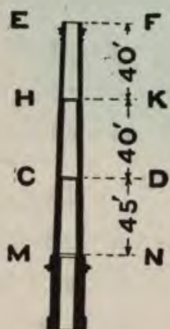

Fig. 112.

joint H K. Next find the maximum stress f_c on the leeward side of the section. It has been shown that the maximum stress f_c is double the mean pressure f caused by the direct weight of the structure above.

Hence,

$$f_c = \frac{2 \times \text{weight of chimney above H K}}{\frac{\pi}{4}(9{\cdot}5^2 - 7{\cdot}25^2) \times 144.}$$

$$= \frac{2 \times 110500}{\pi \times 2{\cdot}25 \times 16{\cdot}75 \times 36} = 51{\cdot}5 \text{ lb. per square inch.}$$

This is well within the limit, and hence the upper section of the chimney is stable so far as the maximum stress is concerned. Now treat the chimney down to the joint C D.

Moment of wind pressure

$$= \frac{\text{P}}{2} \times 9{\cdot}5 \times 80 \times \frac{80}{2} \text{ pounds-feet.}$$

Moment of the weight of chimney above C D about the centre of pressure

$$= \frac{d}{4}\left[110500 + \frac{\pi}{4}\left\{\frac{(9{\cdot}5^2 - 6{\cdot}5^2) + (11^2 - 8^2)}{2}\right\}\right.$$

$$\left.\times 40 \times 100\right] \text{ pounds-feet.}$$

Putting in 11 ft. for d, and equating the two moments above, we get

$$P = 49{\cdot}5 \text{ lb. per square foot.}$$

Also

$$f_c = \frac{2 \times \text{weight}}{\text{area of section}} = \frac{2 \times 275500}{\frac{\pi}{4}(11^2 - 8^2) \times 144} = 85 \text{ lb. per sq. in.}$$

Finally, the moment of the wind pressure about the section M N

$$= \frac{P}{2}\left(\frac{12{\cdot}25 + 8{\cdot}25}{2}\right) \times 125 \times \frac{125}{2} \text{ pounds-feet,}$$

and the moment of the weight about the centre of pressure

$$= \frac{d}{4} [\text{weight of two top sections} + \text{weight of third section}]$$

$$= \frac{12{\cdot}25}{4}\left[275500 + \frac{\pi}{4}\left\{\frac{(11^2 - 7{\cdot}25^2) + 12{\cdot}25^2 - 8{\cdot}5^2)}{2}\right\}\right.$$

$$\left.\times 45 \times 100\right] \text{ pounds-feet}$$

$$= 1640000 \text{ pounds-feet;}$$

and $$40200\,P = 1640000,$$

or $$P = 41 \text{ lb. per square foot.}$$

Further—

$$f_c = 2 \times \frac{\text{weight above section}}{\text{area of section}}$$

$$= \frac{2 \times 536500}{144 \times \frac{\pi}{4}(12{\cdot}25^2 - 8{\cdot}5^2)} = 120 \text{ lb. per square inch.}$$

If the lower courses are set in Portland cement, the above stress will not be excessive, and the chimney will be stable at every section. It is not necessary that any other sections in any particular piece should be tested, because if the lowest is safe, those above will be more so. It is further of no use to test for the tendency to slide at any section, as the

inclination of the resultant to the horizon will never be small enough to permit of a horizontal component that would be great enough to produce sliding.

Stability of Enclosure Wall.

An enclosure wall, 10 ft. high, is built of ordinary brick, and to the section shown in fig. 113. It is 18 in. thick, but the buttresses are $4\frac{1}{2}$ in. thicker than the wall. It is required to investigate the stability of the wall, assuming a certain amount of tension on the windward side.

Take a piece of wall between the centre lines of two alternate buttresses. In this piece there will be a section of plain wall 8 ft. long and 18 in. wide. There remains three bits of wall which, when added together, form a section of 27 in. square, and symmetrical with respect to the neutral axis of flexure.

The mean direct stress f due to the dead weight of the wall

$$= \frac{[(8 \times 1{\cdot}5) + (2{\cdot}25 \times 2{\cdot}25)]\,10 \times w}{[(8 \times 1{\cdot}5) + 2{\cdot}25^2] \times 144} = 7{\cdot}5 \text{ lb. per square inch,}$$

when $w = 110$ lb. per cubic foot of material.

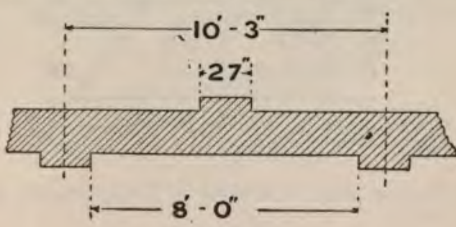

Fig. 113.

The moment of wind pressure about the lowest joint = moment of resistance of that joint;

or,
$$P \times 10{\cdot}25 \times 10 \times 5 \times 12 = \frac{f_b}{h} I$$

$$= f_b \left[\frac{(123 \times 18^3 \times \frac{1}{12}) + (\frac{1}{12} \times 27^4)}{\frac{27}{2}} \right]$$

from which we obtain

$$P = {\cdot}63\, f_b.$$

Now, f_t, the permissible tensile stress, equals $f_b - f$; therefore $f_b = f_t + f = f_t + 7 \cdot 5$;
and for ordinary mortar and brick

$$f_t = 7 \text{ lb. per square inch};$$

also for hydraulic mortar and brick

$$f_t = 14 \text{ lb. per square inch};$$

and for cement and brick

$$f_t = 60 \text{ lb. per square inch}.$$

The best hydraulic mortar used with good bricks, and allowed some months to set, will resist a tensile stress of about 40 lb. per square inch.

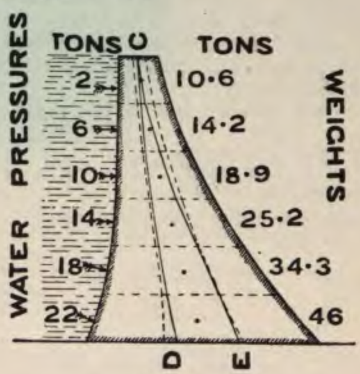

FIG. 114.

Substituting these values for f_t in the preceding equation, we get with

Ordinary mortar,	P = 9·2 lb.	per square foot.
Hydraulic mortar,	P = 13·5 lb.	,, ,,
Cement,	P = 42·5 lb.	,, ,,

STABILITY OF MASONRY DAMS.*

Given the profile of the masonry dam shown in fig. 114, in which 1 cubic foot of material weighs about 160 lb., or ·075 ton, the dam being 10 ft. thick at the top, and increases to 58·5 ft. at the base. It is 72 ft. high measured from the base, and there must be no tension on the water side; neither must the line of resistance pass anywhere outside

* See Appendix.

the middle third of a horizontal section when the reservoir is full or empty. It is required to investigate the stability of the dam.

The section fig. 114 is divided into six parts by horizontal dotted lines 12 ft. apart; and as the lengths of these lines are required, they have been scaled off, and found to be 13·5 ft., 18 ft., 24 ft., 32·5 ft., and 44 ft., taken in order downwards. Consider a piece of the dam 1 ft. long—*i.e.*, measured in a direction perpendicular to the section; then the weight of any portion, included between two adjacent horizontal dotted lines, will be

$$\cdot 075 \times \frac{h_u + h_l}{2} \times 12 \times 1 \text{ tons},$$

where h_u represents the length of the upper horizontal line, and h_l represents the length of the lower horizontal line bounding the piece of section in question. Substituting for the different lengths, we obtain for the weights of the six individual parts, reckoned from the top, 10·6 tons, 14·2 tons, 18·9 tons, 25·2 tons, 34·3 tons, and 46 tons. These loads will act at the centres of gravity of the several portions which are shown by very small circles. The centre of gravity will lie on the line joining the middle points of the upper and lower horizontal lines bounding a piece of the section. Its distance from the upper line is given by the expression

$$12 \cdot \frac{2\,h_l + h_u}{3\,(h_u + h_l)} \text{ ft.*}$$

* Let G be the centre of gravity of the given area, fig. 115, and let A be its area. Also let X be the distance of G from C. The equation to the straight line E F is

$$y = m\,x + c,$$

where $c = \text{C E} = y_1$,

and m = the tangent of the inclination of

$$\text{E F to C D} = \frac{y_2 - y_1}{\text{C D}}.$$

Taking moments about C, we get—

$$\text{A} \times \text{X} = \int_{y_1}^{y_2} y \,.\, d\,x \,.\, x \,. = \int_{0}^{\text{C D}} x\,(m\,x + c)\,d\,x$$

$$= \left[\frac{m\,x^3}{3} + \frac{c\,x^2}{2}\right]_0^{\text{C D}} = \left(\frac{y_2 - y_1}{\text{C D}}\right)\frac{\text{C D}^3}{3} + y_1 \cdot \frac{\text{C D}^2}{2}$$

and

$$\text{X} = \frac{\text{C D}^2\left[\frac{y_2 - y_1}{3} + \frac{y_1}{2}\right]}{\frac{y_1 + y_2}{2} \,.\, \text{C D}} = \text{C D}\left[\frac{2\,y_2 - y_1}{3\,(y_2 + y_1)}\right]$$

Having located the centres of gravity of the six portions of the dam section, the next thing to be done is to find the *line of resistance* when the reservoir is empty. On the right-hand side of fig. 115 the centres of gravity G_1, G_2, and G_3, of the three upper portions of the section are shown, much extended horizontally, for the purpose of more easily distinguishing the several quantities. The horizontal thick lines indicate the assumed joints represented by dotted lines, fig. 114. Let W_1, W_2, W_3, &c., be the weights of the six portions of the section; then the centre of pressure of W_1 and W_2—or, in other words, the point in which the resultant of W_1 and W_2 cuts the second horizontal dotted line, fig.

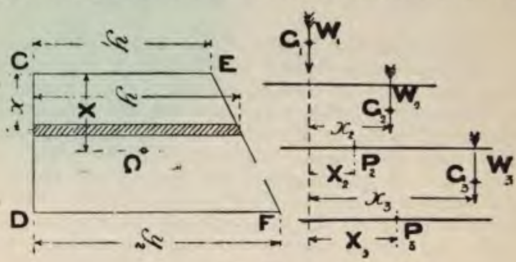

FIG. 115.

114—is at P_2, fig. 115. This point can be determined by taking moments about the vertical through G_1, thus—

$$W_1 \times 0 + W_2 x_2 = (W_1 + W_2) X_2,$$

and
$$X_2 = \frac{W_2 x_2}{W_1 + W_2}.$$

In the same way—

$$X_3 = \frac{W_2 x_2 + W_3 x_3}{W_1 + W_2 + W_3},$$

and
$$X_4 = \frac{W_2 x_2 + W_3 x_3 + W_4 x_4}{W_1 + W_2 + W_3 + W_4},$$

and so on for as many as may be required. Plotting these points in fig. 114, and joining them with a curve, we get the full line curve C D, fig. 114. This is the line of resistance when the reservoir is empty, and it lies wholly within the middle third. The dotted curves indicate the limits of the middle third; therefore the dam is stable when the reservoir is empty.

It is next required to draw the line of resistance when the reservoir is full. In fig. 117 we have the weight W of each piece of the section acting through its centre of gravity, together with the water pressure P over its surface. These two forces will have a resultant R acting through the point in which the two components P and W intersect. The subscript numbers denote the block referred to taken in order from the top. Only a portion of the dam is shown,

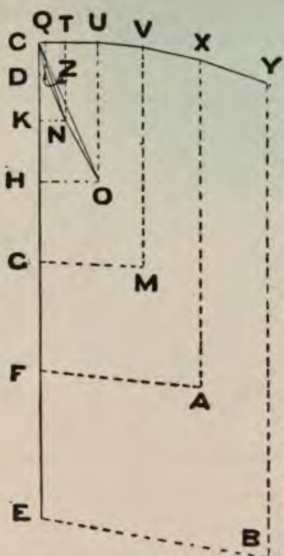

FIG. 116.

it being drawn to a larger scale than fig. 114. It must be understood that the resultants R_1, R_2, &c., fig. 117, are due solely to the weight of, and pressure on, each block separately; for instance, R_3 is the resultant of P_3 and W_3, the pressure of water on the third block and the weight of the third block only, the weight of the blocks above not yet being included. These resultants are very conveniently obtained by setting down the line of weights C E, fig. 116, in which C D represents W_1, D K represents W_2, and so on.

Further, draw C Q to represent P_1 to the same scale; also Q T representing P_2, and so on till X Y represents P_6. Through the points Q, T . . . . Y draw vertical lines (shown dotted); and through D draw D Z parallel to C Q, and K N parallel to C T; also H O parallel to C U, and G M parallel to C V, and F A parallel to C X: and, lastly, E B parallel to C Y The lines C Y, C X, &c., have not been drawn in the figure, as they are unnecessary. The diagonal C Z is the resultant of P_1 and W_1. The diagonal C N is the resultant of P_1, W_1, P_2, and W_2; therefore the diagonal Z N is the resultant of P_2 and W_2.

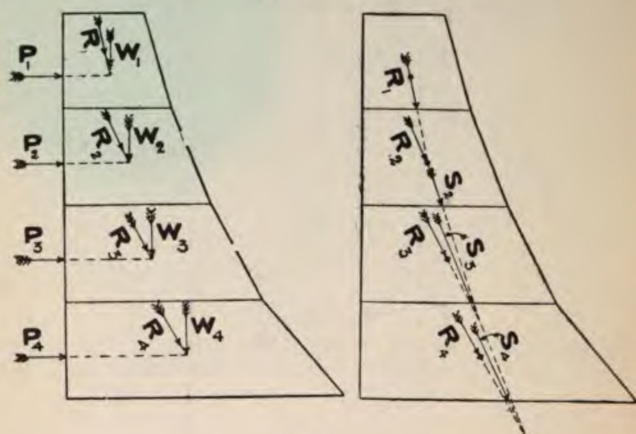

FIG. 117.

In the same way, C O is the resultant of P_1, W_1, P_2, W_2, P_3, and W_3; consequently N O is the resultant of P_3 and W_3. Similarly, O M is the resultant of P_4 and W_4. From this figure the resultants R_1, R_2, &c., can now be put down, as shown in fig. 117. What we are finally seeking is the point in which the resultant of *all* the forces above any joint cuts that joint; for example, the point in which the resultant of P_1, W_1, P_2, W_2, P_3, and W_3 cuts the third horizontal dotted line from the top, fig. 114. Beginning with the top block, the resultant of all the forces acting on that block is R_1, fig. 117. The forces acting on the second block are P_2, W_2, and R_1. The resultant of P_2 and W_2 is R_2, and the

resultant of R_1 and R_2 is S_2, which passes through the point of intersection of R_1 and R_2. The resultant S_2 is C N, fig. 116. The resultant of all the forces acting on the third block—that is, the resultant of P_1, W_1, P_2, W_2, P_3, and W_3—is S_3, fig. 117; that is, the resultant of S_2 and R_3, and acts through the intersection of the lines of action of these two forces. Similarly, S_4 is the resultant of S_3 and R_4. The points in which R_1, S_2, S_3, &c., cut their respective joints are indicated by arrow heads, fig. 117. These points are on the line of resistance when the reservoir is full.

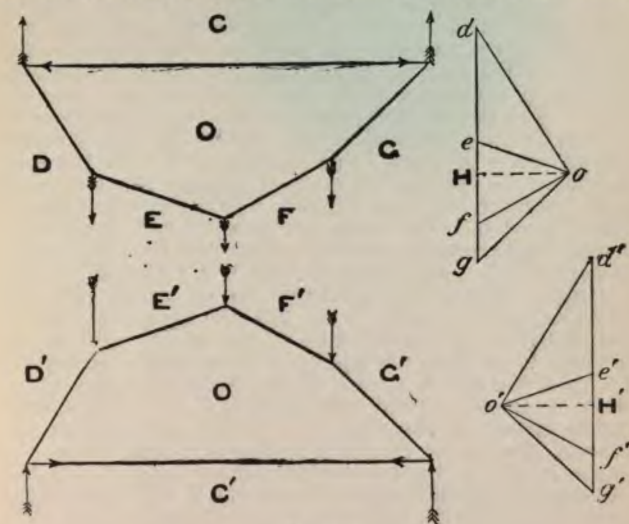

FIG. 118.

These points being joined up, we obtain the full line C E, which, as shown in fig. 114, falls wholly inside the middle third, but touches the limit near the lower end of the dotted line C E. When drawing figs. 114 and 117 the scale should be considerably increased to that in the illustrations; in fact, the larger the scale, the more accurate will be the result. Also the dam should be divided into as many parts as possible by horizontal lines. In the case in question only six parts have been selected, so as not to involve the figure in too many lines to be intelligible.

MASONRY ARCHES.

It has been previously shown that if a linkage be loaded at its joints with vertical loads, and supported at the ends, as shown in fig. 118, the stress in each link is given by the force polygon shown on the right of the figure, and the horizontal component of all the links is the same, and is represented by the horizontal line O H. The member O C is shown as a strut to maintain the two ends apart, but the equilibrium would not be disturbed if the link D O were acted upon at its upper extremity by a single force equal to the resultant of the two there shown, which resultant would act in the same straight line as D O. This would be the direction of the reaction of the pin on the link if the link were simply suspended on the pin without any such member as C O. The vertical components of the reactions, namely, D C and C G, are obtained in any case by drawing through O in the force diagram a line parallel to C O in the link diagram. In this particular case, because the two ends happen to be on the same level, the line through O coincides with O H, or, in other words, the points *c* and H coincide. Then *d c* represents the vertical component D C, and *c g* the component C G. The members D O, E O, F O, and G O are in tension. If now the linkage be turned upside down, as shown in the lower part of fig. 118, the same loads being used as before, the equilibrium of the system will not be disturbed if the links are capable of sustaining compression. The force diagram is shown on the right, that also being turned completely round; the member C_1 O_1 being now in tension instead of compression. In fact, the stresses are reversed throughout, though not altered in magnitude. If the member C_1 O_1 be removed, then there must be applied at the ends a pair of opposite forces which will do the same as the member C_1 O_1. The system of links just described forms an ideal arch, or, more correctly speaking, the line of resistance of it; for on referring to fig. 119, a system of links C, D, E . . . . K are shown loaded at their joints. About the linkage is described with dotted lines, blocks of masonry forming an arch, the joints of which are all perpendicular to the links. Now, it is evident that the arch will behave exactly the same as the linkage in supporting the loads, if we assume that the weights of the blocks are included in the load shown; for take any individual block, say the third from the left. It is shown separated from the rest in the centre of the figure. The thrusts T_1 and T_2, together with the load W_3, are in equilibrium, the same as

in the linkage, and the thrusts T_1 and T_2 act perpendicular to the joints in the masonry, and consequently there can be no tendency of any of the blocks to slide relatively to one another; and further, if the compressive stress is within the safe limit, and the centre of pressure nowhere passes outside the middle third of a joint, the masonry arch is stable, and the line of resistance will coincide with the system of links, in a manner similar to the reservoir dam. As no tie can be inserted between the ends, the reactions R and S must act in the directions alluded to in the previous figure, namely, in $D_1 O_1$ continued, and $G_1 O_1$ continued. It is not necessary that the joints should be perpendicular to the line of resistance, but they must not deviate from that position through an angle greater than the angle of

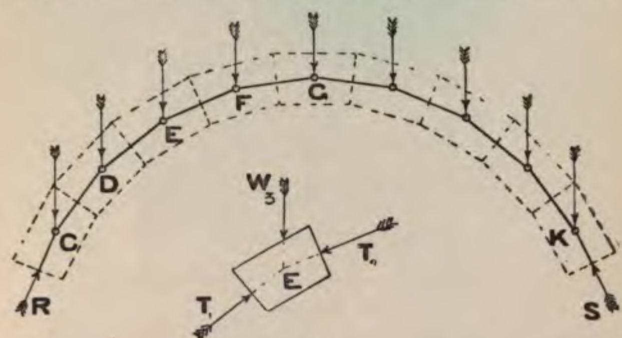

FIG. 119.

repose, which is sometimes called the angle of friction. This is not likely to occur in an arch as usually constructed.

In arches generally, the space between the two curved lines, fig. 120, is called the *arch ring*, while the end stones S are called *springers*. The arch stones *v*, *v* are called *voussoirs*, and the joints between them *bed joints*. The top of the arch ring is called the *crown*, while the *extrados* is the convex surface, and the *intrados* the concave surface, of the arch ring. That portion of the intrados near the springers is sometimes called the *soffit*, and the structure between the arch ring and the roadway is called the *spandrels*. The rise of the arch is shown in the figure as *h*, and the clear span as S. The actual span S_1 is slightly greater, it being the horizontal distance apart of the ends of the line of resistance.

Now, if a body be acted upon by a set of forces, the reactions called into play by those forces, and the internal forces in the body, will be the least possible. This is sometimes stated thus: If forces which are together in equilibrium upon or in a body or structure be classified in two systems, called respectively active and passive forces, which stand to each other in the relation of cause and effect, then will the passive forces be the least which are capable of balancing the active forces, consistently with the physical condition of the body or structure. For the passive forces are caused and brought into play by the application of the active forces to the structure, and they will not increase after the active forces have been balanced by them, and will

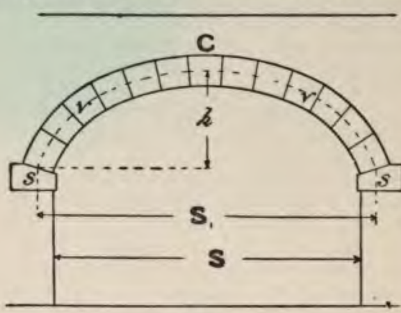

FIG. 120.

therefore not increase beyond the least amount capable of balancing the active forces.

We may therefore conclude that as the force which one member of a structure exerts on another is a minimum with any specified loading, then the horizontal component of the thrust in the arch ring must be a minimum with that loading, and consequently the line of resistance will be that which, consistent with stability, gives the horizontal component a minimum value.

In fig. 121 only that portion of the arch is shown to the left of the point at which the curve of resistance is horizontal; and therefore at this point the thrust in the arch ring equals the horizontal component H. Let the loads on that part of the arch be W_1, W_2, W_3, &c., their sum being W, and let their horizontal distances from the point of application of R be represented by x_1, x_2, x_3, ... and

x respectively. Take moments about the point of application of R, and we have—

$$\mathrm{H}\,h - \mathrm{W}_1\,x_1 - \mathrm{W}_2\,x_2 - \mathrm{W}_3\,x_3 \; . \; . \; . \; . \; \&\mathrm{c.}, = 0;$$

or,
$$\mathrm{H}\,h - \mathrm{W}\,x = 0;$$

that is,
$$\mathrm{H} = \frac{\mathrm{W}\,x}{h},$$

which shows that if the positions of the loads remain the same, and the relative magnitudes of the loads W_1, W_2, W_3,

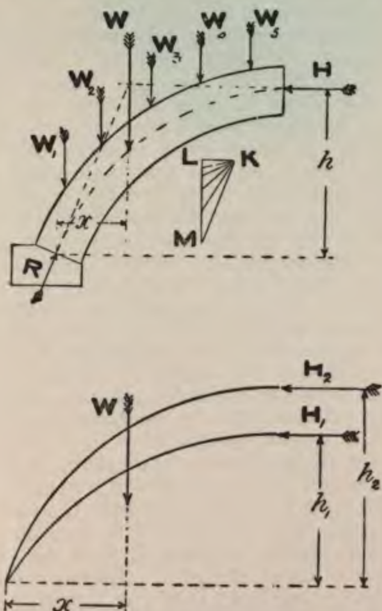

FIG. 121.

&c., do not alter, then the horizontal component varies directly as the sum of the loads.

Let the loads and their positions remain constant; then $\mathrm{W}\,x$ is constant. Now suppose those loads to act successively on the two different linear arches, fig. 121, of heights h_1 and

h_2, the corresponding values of H being H_1 and H_2. From the above equation we have—

$$H\,h = W\,x = \text{constant} = H_1\,h_1 = H_2\,h_2\,;$$

therefore $$H_1 : H_2 :: h_2 : h_1.$$

Thus the horizontal component varies inversely as the height or rise of the arch. In the upper part of fig. 121 produce H to cut the line of action of W, and draw the lines K L and L M to represent the forces H and W respectively. Join K M, and through the point of intersection of H and W draw a line parallel to K M. This will be the line of action of R, and will cut the bearing surface of the springer at the end of the line of resistance. If L M is divided into parts having the ratio W_1, W_2, &c., and each of the points of division be joined to K, then the sides of the funicular polygon are parallel to these joining lines.

In fig. 122 let it be required to draw in the arch ring between the points 3 and 4, so that there shall be a definite, horizontal component H, with the loading given at the top of the figure. The ordinate to the irregularly curved line gives the intensity of the loading at any point. This diagram has been divided by vertical lines into ten parts having the same horizontal breadth, and the area of each of these ten pieces will represent, to a certain scale, the load on that portion of the arch; and consequently the mean ordinate of any one piece gives a measure of that load. A little circle indicates the centre of gravity of each piece, and a vertical dotted line has been drawn through each of them. Bow's notation has been used, and the line of loads a, b, ... m drawn, the point n in it being at present unknown. Set off the horizontal distance H from the load line a, m, drawing the vertical dotted line that must contain the pole. In that line take any pole O_1, and draw in the radiating dotted lines $O_1\,a$, $O_1\,b$, &c., and afterwards complete the funicular polygon 1, 2 in the usual manner, including the closing line 1, 2. Through O_1 draw $O_1\,n$ parallel to the closing line; then a, n represents the vertical component of the reaction at 4, and m, n, the vertical component of the reaction at 3, the same as in a beam problem. Now, we took *any* pole O_1, and obtained the funicular polygon 1, 2. If any other pole had been taken, we should have obtained another polygon different to 1, 2; but as we have not altered the position of the loads, we shall get the reactions in the vertical direction the same as before, namely, a, n and m, n. That is to say, wherever the pole O_1 is

selected, the point n remains unaltered in position. Now, the full line $O_1\ n$ is parallel to the closing line 1, 2. The closing line of the funicular polygon which we require to draw is 3, 4, and as the point n remains unaltered for all poles, and as the new line corresponding to $O_1\ n$ must pass through n, draw $n\ O_2$ parallel to 3, 4. The point where it cuts the vertical through O_1 is the new pole O_2. Draw in

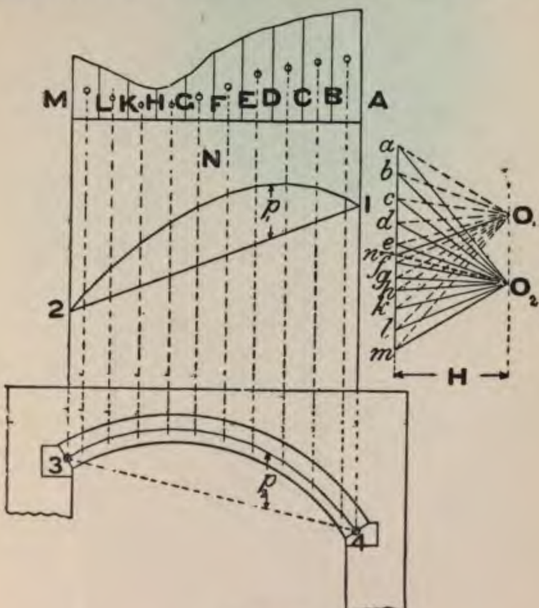

FIG. 122.

the new radial lines and a new funicular polygon, which if started at 4 will end at 3, this polygon being the line of resistance of the arch, sometimes called the linear arch. It will be observed that the lower funicular polygon is simply the upper one distorted by pure shear in a vertical direction, the angle of distortion being $O_1\ n\ O_2$, and consequently the vertical ordinates of the two are the same, or $p_1 = p_2$. This might have been anticipated from previous work

dealing with the graphic determination of bending moment, for, wherever the pole is selected, the vertical ordinate is a measure of the bending moment; and as the loads are unaltered, the bending moment cannot vary; and as the pole distances in both cases equal H, the vertical ordinate of the funicular polygon must be the same. In this way the lower polygon might have been plotted by simply transferring the vertical ordinates from the upper polygon to the line 3, 4

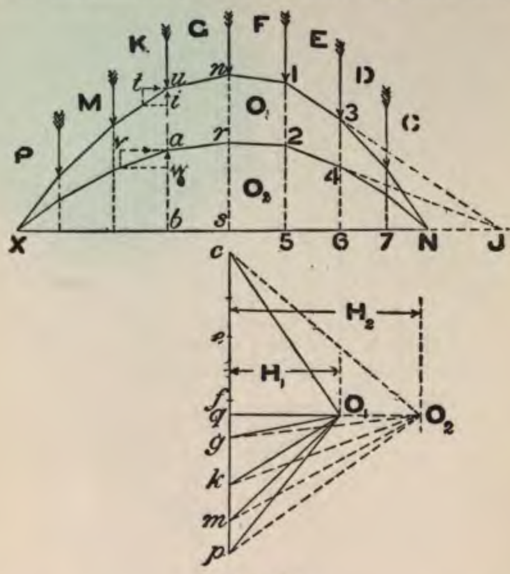

Fig. 123.

As a preliminary to the next problem, it will now be shown that the ratio

$$\frac{H_1}{H_2} = \frac{h_2}{h_1},$$

not only holds good for the points in two linear arches at which those curves are horizontal, but that the same is true for any other pair of points in which a vertical line cuts two funicular polygons which terminate in the same points, these points being in the same horizontal line. In fig. 123

there are drawn two funicular polygons through X and N, corresponding to the two horizontal components H_1 and H_2, represented by $q\,O_1$ and $q\,O_2$. The closing line X N being the same for both polygons, the lines $q\,O_1$ and $q\,O_2$ will coincide. Let $n\,s = h_1$, and $r\,s = h_2$. Consider the points

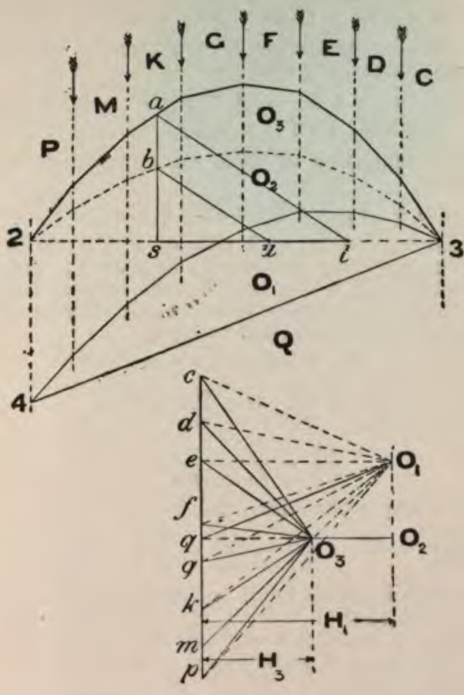

FIG. 124.

of application of the load K G to each of the arches—*i.e.*, the points u and a respectively. Using Bow's notation, the thrust $k\,O_1$ in the member K O_1 can be resolved into its two rectangular components, $t\,n$ horizontally and $i\,u$ vertically. These we may designate H_1 and V_1 respectively.

For the equilibrium of the piece of arch between u and N we must have, by the second law of equilibrium—

Moment of H_1 about N + moment of V_1 about N = sum of moments of forces K G, G F, F E, E D, and D C about N; or—

$$H_1 \times u b) + (V_1 \times b\,N) = (K\,G \times b\,N) + (G\,F \times S\,N) + (F\,E \times 5\,N) + (E\,D \times N\,6) + (D\,C \times 7\,N).$$

And with similar treatment of the point a in the lower arch, using suffix 2 instead of 1—

$$(H_2 \times a b) + (V_2 \times b\,N) = (K\,G \times b\,N) + (G\,F \times S\,N) + (F\,E \times 5\,N) + (E\,D \times 6\,N) + (D\,C \times 7\,N).$$

The right-hand sides of these equations are the same; consequently

$$(H_1 \times u b) + (V_1 \times b\,N) = (H_2 \times a b) + (V_2 \times b\,N).$$

The latter bracket on each side of this equation is the same, for, on referring to the pole diagram V_1, the vertical component of K O_1 is $k\,q$; and V_2, the vertical component of K O_2, is also $k\,q$; hence these equals may be removed from the equation, which then becomes

$$H_1 \times u b = H_2 \times a b,$$

or
$$\frac{H_1}{H_2} = \frac{a b}{u b}, \text{ but } \frac{H_1}{H_2} = \frac{h_2}{h_1};$$

therefore
$$\frac{a b}{u b} = \frac{h_2}{h_1} = \frac{r s}{n s}.$$

Similarly for any other joints, thus:

$$\frac{H_1}{H_2} = \frac{h_2}{h_1} = \frac{r s}{n s} = \frac{a b}{u b} = \frac{2, 5}{1, 5} = \frac{4, 6}{3, 6}, \text{ \&c.}$$

And the same, of course, holds between the joints. It may also be noticed that as 2, 5 : 1, 5 : : 4, 6 : 3, 6, and, therefore, the lines 1, 3 and 2, 4 will intersect X N in the same point J. The same relation exists with any other pair of corresponding lines.

Problem: Given the springer points 2 and 3, fig. 124, and a third point a in the linear arch, loaded as shown with any loads, it is required to draw the linear arch.

Produce the lines of action of the loads as shown by dotted lines, including lines through the points 2 and 3. Set down the line of loads $c\,p$, and take *any* pole O_1. Join this pole to the points $c, d, e,$ . . . . p, and draw the funicular polygon 3, 4 with sides parallel to the polar lines, as usual.

Draw the closing line 3, 4, and then the companion parallel line $O_1 q$. The point q divides the line of loads into two parts, such that $c\,q$ represents the vertical component of the reaction at 3, while $p\,q$ represents the vertical component of the reaction at 4, exactly the same as in the bending moment problems. In these latter, it will be remembered that the vertical reactions were thus found *wherever* the pole O_1 was

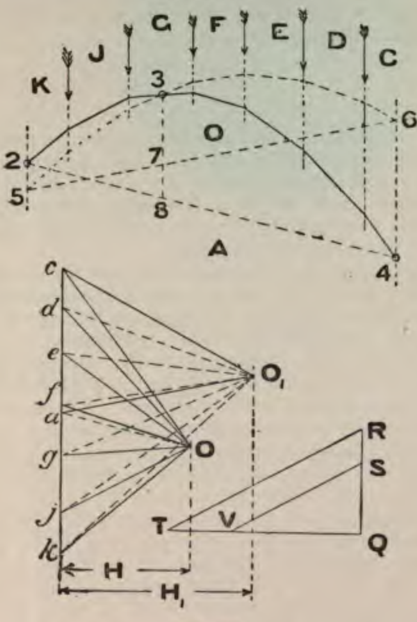

FIG. 125.

taken; and as the ordinate cut-off by the funicular polygon was the same as long as the horizontal pole distance remained constant, if we draw $q\,O_2$ through q, parallel to 2, 3, and equal to H_1, we can then draw another funicular polygon $3\,b\,2$ (shown dotted) with sides parallel to the radial lines $O_2\,c$, $O_2\,d$, &c., which must pass through 2, and whose vertical intercepts are equal to those of the polygon 3, 4.

But the polygon $2b3$ does not pass through the point a as desired. Using the result of the discussion immediately preceding this problem, we have

$$bs : as :: H_1 : H_2$$

a being a point on the required funicular polygon. Set off st equal to qO_2, and join at; then through b draw bu parallel to at, cutting 2, 3 in u. Then su is the required H_3, and the distance qO_3 is set off equal to su. This is evident, for by similar triangles

$$\frac{bs}{as} = \frac{su}{st} = \frac{H_3}{H_1}.$$

A third polygon is drawn, whose sides are parallel to the radial lines $O_3 c$, $O_3 d$, &c., and this polygon will pass through a and 2, the horizontal component being H_3.

In an actual arch, in which the loads are continuous, the polygon will be a continual curve, which will pass through the corners of the given polygon.

If the springing points 2 and 3 are not on the same level, the process is the same, but it must be remembered that st and su are still on the line 2, 3, and not horizontal, as shown. After having obtained the line of resistance, or linear arch, with maximum and minimum loads, symmetrical curves must then be drawn, so that these lines of resistance shall be altogether included between them. These symmetrical curves are the limits of the middle third of the arch ring.

A very short method upon the principles here set forth of drawing the linear arch through three given points is shown in fig. 125. The points are 2, 3, 4. The line of loads ck being drawn to some convenient scale, and *any* pole O_1 assumed, the funicular polygon 5, 3, 6 is drawn, commencing with the member GO through the given point 3. The line aO_1 is then drawn parallel to 5, 6, and ca, ak are the vertical components of the end reactions. Draw 3, 7, 8 through 3 vertically, and join the two given points 2, 4; this will be the closing line of the required linear arch. Through a draw aO parallel to 2, 4, and take the point O in it such that $H : H_1 :: 3, 7, : 3, 3$. This is easily done thus: Set down QS and QR equal to 3, 7 and 3. 8 respectively; also QT equal to H_1. Join RT, and through S draw SV parallel to RT, cutting TQ in V Then QV is the required H. Set off H to the right of the line of loads, and the vertical through its extremity cuts aO in O. This is the required pole. After drawing the

radial lines, the linear arch 2, 3, 4 is then drawn, commencing with either of the points 2, 3, or 4. In general, with granite voussoirs, if ρ be the radius of the circle which touches the topmost point of the intrados, and the inside faces of the springer, then depth of keystone at crown

$$= \cdot 25 \sqrt{\rho + \cdot 5 \text{ span}} + \cdot 2 \text{ ft.}$$

For ordinary stone arch work multiply this result by $\frac{9}{8}$, and for brickwork multiply by $\frac{4}{3}$.

The depth of springers is the same as depth of keystone, except for excessive spans. In small arches the depth of keystone is often taken as

$$\sqrt{1 \cdot 2 \text{ radius of curvature of soffit at crown}}$$

for single arches, and for a series of arches the constant 1·2 is replaced by 1·7.

CHAPTER XXVI.

The Principle of Work and Miscellaneous Problems.

Let a force F be applied at the end of a piece of elastic material, whose transverse sectional area is A square inches and uniform, while its length is l inches, and Young's modulus of elasticity is E. The sign of F is positive if tensile, and negative if compressive, and any extension due to the force F will have the same sign as F.

The elongation of the material under the action of the load F = strain × length; but

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\frac{F}{A}}{\text{strain}};$$

$$\therefore \text{strain} = \frac{F}{AE},$$

and consequently the extension

$$= \frac{F\,l}{A\,E}.$$

The work done in steadily straining the material = average force × extension

$$= \frac{F}{2} \times \frac{F\,l}{A\,E} = \frac{F^2}{2} \times \frac{l}{A\,E} = \frac{F^2}{2} \times m \quad . \quad . \quad . \quad (a)$$

where

$$m = \frac{l}{A\,E} = \text{the extension produced by a load of 1 lb.}$$

In equation (a) it is evident the average load is $\frac{F}{2}$, because the load is gradually and steadily applied without shock of any kind. We may put the matter in another way, thus: Let a scale pan be attached to the material, and let some finely-divided substance, such as sand or shot, be steadily poured into the pan; the material will stretch as the load increases, and as the load is uniformly increased from zero to F, the average must be

$$\frac{F + O}{2} = \frac{F}{2}.$$

It may be argued that the putting on of a load is a different operation to the pouring in of a finely-divided substance such as shot; but upon a little consideration it will be seen that the two are identical. Take the case of putting a weight on an ordinary spring balance; say the weight is 10 lb., and is held in the hands. The spring stretches as the load is applied, and when the balance indicates, say, 6 lb., the remaining 4 lb. is upheld by the hands. Hence, at every point in the elongation, some part of the load is supported by the hands, beginning at zero elongation with supporting the whole of it, and gradually allowing the balance to take more and more of the weight, until the hands support none of it at maximum elongation. This is what takes place when a load is *steadily* applied to any elastic structure.

Now, from the principle of the conservation of energy, or what is sometimes called the principle of work, the work done upon a structure by all the external forces equals the work done in deforming the members of the structure. This principle will be applied to solve some of the problems which follow.

Consider the hinged frame D B C, fig. 126, in which D B is 5 ft., B C 7 ft., and D C 10 ft.; also the sectional areas of D B and B C are 1 and 3 square inches respectively. Let

the points D and C be maintained in their respective positions—*i.e.*, the member D C must be absolutely rigid. For convenience in calculation, let suffix 1 relate to D B, and suffix 2 to B C. It is required to find the vertical deflection of the point B. Work done in extending

$$DB = \frac{F_1{}^2}{2} m_1,$$

where F_1 represents the total stress in D B. [Vide equation (α).]

In the same way, the work done in extending

$$BC = \frac{F_2{}^2}{2} m_2,$$

the work done by the external force

$$W = \frac{W}{2} \delta,$$

where δ = the vertical deflection of B. Then, as the work done upon the frame by the external forces equals the work done in deforming the members of the frame,

$$\frac{W}{2} \delta = \frac{F_1{}^2}{2} m_1 + \frac{F_2{}^2}{2} m_2 \quad . \quad . \quad . \quad . \quad (\beta)$$

$$m_1 = \frac{l_1}{E A_1} = \frac{5 \times 12}{12000 \times 1} = \tfrac{1}{200}$$

$$m_2 = \frac{l_2}{E A_2} = \frac{7 \times 12}{12000 \times 3} = \tfrac{7}{3000}.$$

The total stresses F_1 and F_2 must be found by ordinary statical methods. The triangle of forces at the point B is shown in fig. 126, and as the triangle of the framework and the triangle of forces are similar,

$$\frac{F_1}{W} = \frac{DB}{DC} = \tfrac{5}{10},$$

and

$$\frac{F_2}{W} = \frac{BC}{DC} = \tfrac{7}{10}.$$

Inserting these values in equation (β), we have—

$$\delta = \frac{F_1{}^2 m_1 + F_2{}^2 m_2}{W}$$

$$= \frac{\dfrac{2{\cdot}5^2}{200} + \dfrac{3{\cdot}5^2 \times 7}{3000}}{5}$$

$$= {\cdot}012 \text{ in.}$$

The same result might have been obtained thus :

Extension of D B $= F_1 m_1 = 2{\cdot}5 \times \frac{1}{200} = {\cdot}0125$ in.

Extension of B C $= -F_2 m_2 = -3{\cdot}5 \times \frac{7}{3000} = -{\cdot}0082$ in.

In the upper part of fig. 126, drawn to a much larger scale, are the extensions B G and B Q, representing ·0125 in., and – ·0082 in. respectively. Through G draw G N perpendicular to B G, and through Q draw Q N perpendicular to Q B, cutting G N in N. The point B is deflected by the load W to the point N, through a vertical distance B R, and the horizontal distance R N. The former, if measured off, equals ·012 in., as found above, and the latter equals ·0051 in.

Another and useful method of finding R N, the horizontal deflection, is as follows : Put on a force H, acting at B in a horizontal direction, fig. 127, from left to right, in addition to the load W. The work done by H on the structure will be—

$$\frac{H}{2} \times \delta_h,$$

where δ_h = horizontal deflection.

The work done by H on the structure must equal that done in deforming the individual members of the structure, and it is this quantity we must next find.

Let two forces, P and Q, act in the same direction and together upon a piece of material, say along its axis. The work done

$$= \tfrac{1}{2}(P + Q)^2 m = \tfrac{1}{2}(P + Q)(P + Q)\, m$$

$$= \frac{P}{2}(P + Q)\, m + \frac{Q}{2}(P + Q)\, m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

$$= \text{work done by P} + \text{work done by Q}.$$

Hence the work done by one of a number of forces on a member equals the product of m and half the force in question into the sum of all the forces acting on the member

Referring to fig. 127, and considering the stresses in D B and B C due to H alone, we find, after drawing the triangle of forces X Y Z, the angle

$$Y = \frac{\pi}{2} - D,$$

the angle

$$X = \frac{\pi}{2} - C,$$

and the angle $\quad Z = C + D.$

And further, because in any triangle the sides are proportional to the sines of the opposite angles, then

$$\frac{H}{\sin Z} = \frac{F_1{}^1}{\sin X} = \frac{F_2{}^1}{Y}$$

and putting in the equivalents of the angles, we have

$$\frac{H}{\sin (C + D)} = \frac{F_1{}^1}{\sin\left(\frac{\pi}{2} - C\right)} = \frac{F_2{}^1}{\sin\left(\frac{\pi}{2} - D\right)}$$

or,

$$\frac{H}{\sin B} = \frac{F_1{}^1}{\cos C} = \frac{F_2{}^1}{\cos D}.$$

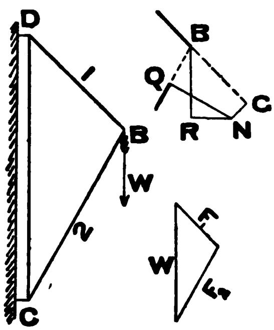

FIG. 126.

Now, from trigonometry we get

$$\sin B = \frac{2}{DB \times BC}\sqrt{S\,(S - DB)\,(S - BC)\,(S - DC)}$$

where S = half the sum of the three sides.
In this way we find

$$\sin B = \frac{2}{5 \times 7}\sqrt{11\,(11 - 5)\,(11 - 7)\,(11 - 10)} = {\cdot}93\,;$$

also $$\cos C = \frac{DC^2 + CB^2 - DB^2}{2 \times DC \times CB} = {\cdot}88,$$

and $$\cos D = \frac{DC^2 + DB^2 - BC^2}{2 \times DC \times DB} = {\cdot}76.$$

Using these values in the above equation, we have

$$F_1{}^1 = \cdot 95\,H, \text{ and } F_2{}^1 = \cdot 82\,H.$$

It has also been previously found that the total stress in DB and CB due to W alone are $F_1 = 2{\cdot}5$ tons and $F_2 = 3{\cdot}5$ tons respectively. Now, in DB we have stress due to both loads W and $H = F_1 + F_1{}^1 = 2{\cdot}5 + \cdot 95\,H$, and in BC we have $-F_2 + F_2{}^1 = -3{\cdot}5 + \cdot 82\,H$; the negative sign showing that W tends to shorten BC, while H tends to lengthen it.

Now, from equation (γ) we get—Work done by H on member DB equals the continued product of m_1, half the stress due to H, and the sum of the stresses due to H and W,

$$= \frac{F_1{}^1}{2}(F_1 + F_1{}^1)\,m_1 = \frac{\cdot 95\,H}{2}(2{\cdot}5 + \cdot 95\,H);$$

and the work done on BC by H

$$= \frac{F_2{}^1}{2}(-F_2 + F_2{}^1)\,m_2 = \frac{\cdot 82\,H}{2}(-3{\cdot}5 + \cdot 82\,H);$$

and from the general equation of work we have

$$\frac{H}{2}\,\delta_h = \frac{\cdot 95\,H}{2}(2{\cdot}5 + \cdot 95\,H)\,m_1 + \frac{\cdot 82\,H}{2}(-3{\cdot}5 + \cdot 82\,H)\,m_2;$$

or, $\delta_h = \cdot 95\,(2{\cdot}5 + \cdot 95\,H)\,m_1 + \cdot 82\,(\cdot 82\,H - 3{\cdot}5)\,m_2$.

Now we require to find the horizontal deflection of B with W alone—that is, when H is zero; hence in the last equation put H = O, and there remains

$$\delta_h = \frac{(\cdot 95 \times 2{\cdot}5)}{200} - \frac{(\cdot 82 \times 3{\cdot}5) \times 7}{3000} = \cdot 0051 \text{ in.}$$

Further, let DC still remain rigid, and let a bar, 1 square inch in section, connect B to T, of the same material as that in DB and BC. The structure now contains one more member than is required for statical equilibrium, and it is required to determine the stresses in the members BD, BC, and BT.

Let the actual stress in BT be represented by H; then we may remove TB and replace it by its stress H, as shown in fig. 127, on the right. Using the same notation as in the previous example, we must have—Stress in DB is F_1 due to W alone, without H, and $-F_1{}^1$ due to H alone, without W.

Similarly, stress in BC is $-F_2$ due to W alone, and $-F_2{}^1$ due to H alone, the negative signs indicating compression.

Now, the member B T is one of a system which is a conservative system of members, and consequently the work done by any one member is stored up in the others ; or in other words, the work of the whole system remains constant, and is unaffected by any individual member. Hence the work done by any member on all the members of the system, including itself, must be zero. It may be seen more easily, thus : The work done *by* a member is that amount which it gives up to others, and is consequently negative from that point of view. The work done *on* the other members is received by them, and is consequently positive. Further, the work done *by* the one member equals the work done *on* the remainder of them, and the total work done on the system of members (*i.e.*, on all of them) = *minus* that given out by the one *plus* that received by the remainder, = zero. Therefore, we may state that the work done by a redundant member on the whole structure (including itself) is zero. It will be useful to tabulate the results as they are obtained, thus :—

Member.	$m = \frac{l}{AE}$	Stress due to W alone.	Stress due to H alone.	Elongation = m × total stress.	Work due to H alone = stress due to H alone × elongation × $\frac{1}{2}$.
D B	$\frac{1}{200}$	2·5	$-$ ·95 H	·0125 $-$ ·00475 H	($-$ ·0119 H + ·0045 H^2) × $\frac{1}{2}$
B C	$\frac{7}{3000}$	$-$ 3·5	$-$ ·82 H	$-$ ·0082 $-$ ·0019 H	(·0067 H + ·00156 H^2) × $\frac{1}{2}$
B T	$\frac{13}{4000}$	0	H	·0032 H.	·0032 H^2 × $\frac{1}{2}$

The length of the B T is 3 ft. 3 in. ; and, consequently,

$$m = \frac{13}{4000} = \frac{l}{AE}.$$

Also in Column III., the stress in B T due to W alone is zero, as the stress in B T does not now exist as a stress, it being replaced by an external force H. The negative signs in Column IV. indicate that the stresses in D B and B C due to H alone are compressive. The total stress spoken of in Column V. is the stress in III. added to that in IV. This sum is multiplied by the corresponding value in II., and placed in V. As we require the work done by H alone on the whole structure, we must multiply the quantities in V. by those in IV., and place the products in Column VI. The work done by H on the whole structure is the sum of

the quantities in Column VI., and from above this sum must be zero. Therefore, sum of quantities in Column VI. $= \cdot 00926\,H^2 - \cdot 0052\,H = 0$, or $H = \cdot 562$ ton.

Having obtained the numerical value of H, we may then draw the polygon of forces, and thus deduce F_1 and F_2. In this way $F_1 = 2$ tons, $F_2 = 4{\cdot}05$ tons, and the effect of introducing this extra member was not to materially strengthen the structure as a whole, but rather to weaken it; for while it relieved D B of half a ton, it added about half a ton to B C; and if this latter member were in the first place only designed for a stress of 3·5 tons, the

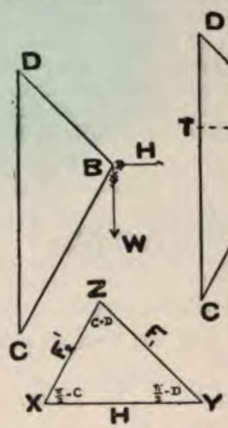

FIG. 127.

introduction of the member B T would tend to destroy the structure by overloading B C.

Another, but slightly different, method may be adopted to find the stresses in a structure which contains redundant members. In connection with the masonry arch, it was shown that the internal forces in a structure brought into play by the application of external forces were always a minimum. In the case of a structure made up of elastic material with hinged joints, the internal forces are the stresses in the members; and consequently each of these is a minimum. The work done against these stresses is given by the sum of all the quantities, $\frac{1}{2}\,F^2\,m$; consequently,

the work done is a minimum, because F is a minimum. If the work of deformation is a minimum, it is shown in the Differential Calculus that the first differential coefficient of the work done is zero. Of all the quantities that come into the expression for work done, the only ones which are independent variables are the stresses in the redundant members; for if they did not exist, the remaining stresses are all determined, and when the redundant members are introduced, the stresses in all members are affected by the redundant members. Hence differentiate the expression for the work of deformation with respect to each of the stresses in the redundant members, and equate each to zero. This will give as many equations as there are redundant or superfluous members, from which their stresses may be determined.

Applying this to the structure, fig. 127, we obtain for the work done in deforming the structure from the previous table, the sum of all the quantities—(stress due to W + stress due to H) $\times \dfrac{\text{elongation}}{2}$. Thus work done by all forces on structure

$$= (2{\cdot}5 - {\cdot}95\text{H})({\cdot}0125 - {\cdot}00475\text{H}) + (3{\cdot}5 + {\cdot}82\text{H})({\cdot}0082 + {\cdot}0019\text{H}) + {\cdot}0032\,\text{H}^2 = \text{W, say.}$$

Then $$\text{O} = \frac{d\text{W}}{d\text{H}} = (2{\cdot}5 - {\cdot}95\,\text{H})(-{\cdot}00475)$$
$$+ ({\cdot}0125 - {\cdot}00475\,\text{H})(-{\cdot}95) + (3{\cdot}5 + {\cdot}82\,\text{H}) \times {\cdot}0019 + {\cdot}82\,({\cdot}0082 + {\cdot}0019\,\text{H}) + {\cdot}0064\,\text{H},$$

from which we obtain

$${\cdot}01852\,\text{H} - {\cdot}0104 = \text{O};$$

or $$\text{H} = {\cdot}56 \text{ ton},$$

the same as obtained previously.

In the last example only one superfluous bar or member was introduced. We will now proceed to investigate the problem in its general sense with any number of redundant members.

A series of articles on this subject appeared in *Engineering* during the latter part of 1894 and the early portion of 1895, by Mr. H. M. Martin, Wh.Sc., and numerous articles have appeared in the *Engineer* between 1886 and 1895 inclusive from the pen of Mr. Max Am Ende, M.I.C.E., all of which are well worth reading. The method of treatment in what immediately follows is that adopted, and so well described, by Mr. Max Am Ende.

The number of superfluous members in a structure is easily determined, for where there are none

$$m = 2j - 3,*$$

where m is the number of members and j the number of joints or nodes. In fig. 128 there are 9 members and 6 joints; therefore, by the above equation, there are no superfluous members. In the figure the members are numbered, and a pair of equal and opposite forces F F are applied at the points X and Y.

Consider the stress in any member, say the rth produced by the forces F. Let S_r denote this stress and l_r the length

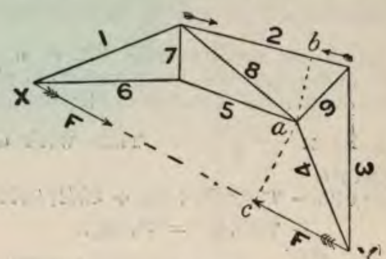

FIG. 128.

of that member, while δl_r will represent its extension. At the same time, let A_r be the sectional area and E the modulus of elasticity. In the figure, we may replace any member, say number 2, by the stress in that member. Taking moments about the point a, we get

$$F \times ca - S_2 \times ab = o,$$

or $$S_2 = \frac{ac}{ab} F = c_2 F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma_1)$$

where c_2 represents the fraction $ac \div ab$. In the same way

$$S_3 = c_3 F, \text{ and } S_r = c_r F.$$

The coefficient c may be always found from statical considerations alone, and it represents the stress in the member due to an applied force of unit magnitude in the place of F.

If the structure be elastic, the points X and Y will approach each other, and the different members of the

* See Appendix.

structure will be deformed. If δl_F denote the alteration of length X Y, then the work done by the forces F on the structure will be

$$\frac{F}{2} \times \delta l_F,$$

and will be positive, because the points X and Y move in the direction of the forces. The work done against the stress in any member

$$= \frac{S}{2} \times \delta l = \frac{c F \delta l}{2};$$

and the work done in deforming the members of the structure by the force F will be

$$\tfrac{1}{2}[c_1 F \delta l_1 + c_2 F \delta l_2 + c_3 F \delta l_3 + \&c.]$$

As all these quantities of work are done by the stresses in the several members, the whole will be negative in sign relative to the work done by the forces F, because movement takes place in a direction opposite to that in which the stresses act.*

The principle of work says that *the work done by the external forces equals that done against the internal stresses;* or, if the proper signs be attached to show how the work is done relative to the forces F, *the sum of the external and internal work is zero*; or

$$O = \frac{F}{2}\delta l_F - \frac{F}{2}[c_1 \delta l_1 + c_2 \delta l_2 + c_3 \delta l_3 + \&c.],$$

and $$O = \delta l_F - c_1 \delta l_1 - c_2 \delta l_2 - c_3 \delta l_3 - \&c.$$

If now we replace the forces F in fig. 128 by a superfluous member, as in fig. 129 (the structure being hinged at X and free at Y), the work done by the internal stress in this member will be negative, because the point Y

* There should be no difficulty in determining the sign of the work done, for the work done by the forces F is really energy given to the structure by the agent which is acting on the structure with the forces F. This same energy is stored up in the elastic structure—*i.e.*, the structure receives it from the agent. Now, the sign of the energy will depend upon the point of view of looking at it—*i.e.*, whether from the point of view of the recipient or of the donor. Looking at the question in this way, it is easy to see that the work done by the external forces equals the work done in deforming the members of the structure.

moves in the opposite direction to that in which the stress is applied. The above equation will then become—

$$O = \delta l_a + c_1{}^a \delta l_1 + c_2{}^a \delta l_2 + c_3{}^a \delta l_3 + c_4{}^a \delta l_4 + \&c. \quad (\delta)$$

where a is not an index, but simply implies that the coefficient c refers to the member denoted by the suffix in relation to the superfluous member X Y, whose stress is F^a; thus $c_2{}^a$ represents the coefficient

$$c = \frac{a\,c}{a\,b}$$

in fig. 128. In the same way $c_2{}^b$ would represent the same coefficient connecting a second superfluous bar with member number 2—that is, it is the stress in member 2 produced by unit stress in the superfluous member b.

Now let the structure, fig. 129, be acted upon by the external forces P, Q, &c., and let there be superfluous bars

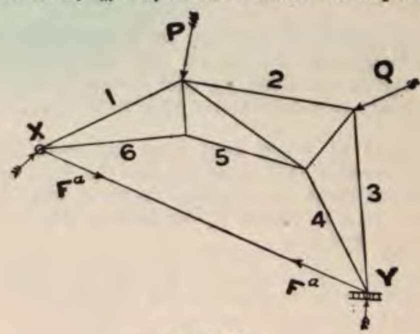

Fig. 129.

in it (not shown), whose stresses are represented by F^a, F^b, F^c, &c. Let R represent the stress in any member produced by the external forces, without any superfluous member being in the structure.

Then the whole stress in any member, say number 1, due to the external forces and the redundant members together, will be

$$S_1 = R_1 + S_1{}^a + S_1{}^b + S_1{}^c + \&c.,$$

where the letters a. b, &c., denote the superfluous members producing those stresses.

But it has been shown that

$$S_1{}^a = c_1{}^a F^a, \quad S_1{}^b = c_1{}^b F^b, \text{ and } S_1{}^c = c_1{}^c F^c;$$

therefore

$$\left.\begin{array}{l} S_1 = R_1 + c_1{}^a F^a + c_1{}^b F^b + c_1{}^c F^c + \&c. \\ \text{Similarly,} \\ S_2 = R_2 + c_2{}^a F^a + c_2{}^b F^b + c_2{}^c F^c + \&c. \\ \text{and} \\ S_3 = R_3 + c_3{}^a F^a + c_3{}^b F^b + c_3{}^c F^c + \&c. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (\epsilon)$$

and so on for all the members.

It has also been shown that the extension of any member δl

$$\begin{aligned} &= \text{strain} \times \text{original length of member} \\ &= \frac{\text{stress intensity}}{E} \times l \\ &= \frac{\text{total stress}}{\text{sectional area}} \times \frac{l}{E} \\ &= S \times \frac{l}{AE} = S\,m, \text{ say} \quad . \quad . \quad . \quad . \quad . \quad (\zeta) \end{aligned}$$

For every superfluous member we shall have an equation similar in form to (δ), and, therefore, in the present instance we must have the equations—

$$\left.\begin{array}{l} O = \delta l_a + c_1{}^a\,\delta l_1 + c_2{}^a\,\delta l_2 + c_3{}^a\,\delta l_3 + \&c. \\ O = \delta l_b + c_1{}^b\,\delta l_1 + c_2{}^b\,\delta l_2 + c_3{}^b\,\delta l_3 + \&c. \\ O = \delta l_c + c_1{}^c\,\delta l_1 + c_2{}^c\,\delta l_2 + c_3{}^c\,\delta l_3 + \&c. \end{array}\right\} \quad . \quad . \quad (\eta)$$

And so on for every superfluous member. Substituting from (ζ) in (η), we get—

$$O = S_a\,m_a + c_1{}^a\,S_1\,m_1 + c_2{}^a\,S_2\,m_2 + c_3{}^a\,S_3\,m_3 + \&c.$$

$$O = S_b\,m_b + c_1{}^b\,S_1\,m_1 + c_2{}^b\,S_2\,m_2 + c_3{}^b\,S_3\,m_3 + \&c.$$

$$O = S_c\,m_c + c_1{}^c\,S_1\,m_1 + c_2{}^c\,S_2\,m_2 + c_3{}^c\,S_3\,m_3 + \&c.$$

And further substituting from equation (ϵ), we have—

$$\left.\begin{array}{l} O = S_a\,m_a + c_1{}^a\,m_1\,(R_1 + c_1{}^a S_a + c_1{}^b S_b + c_1{}^c S_c + \&c.) \\ \qquad\qquad + c_2{}^a\,m_2\,(R_2 + c_2{}^a S_a + c_2{}^b S_b + c_2{}^c S_c + \&c.) \\ \qquad\qquad + c_3{}^a\,m_3\,(R_3 + c_3{}^a S_a + c_3{}^b S_b + c_3{}^c S_c + \&c.) \\ \qquad\qquad\qquad + \&c. \\ \text{Similarly,} \\ O = S_b\,m_b + c_1{}^b\,m_1\,(R_1 + c_1{}^a S_a + c_1{}^b S_b + c_1{}^c S_c + \&c.) \\ \qquad\qquad + c_2{}^b\,m_2\,(R_2 + c_2{}^a S_a + c_2{}^b S_b + c_2{}^c S_c + \&c.) \\ \qquad\qquad + c_3{}^b\,m_3\,(R_3 + c_3{}^a S_a + c_3{}^b S_b + c_3{}^c S_c + \&c.) \\ \qquad\qquad\qquad + \&c. \end{array}\right\} (\theta)$$

and so on for as many equations as there are superfluous bars. If there is only one superfluous bar A, then S_b, S_c, &c., will be zero; and the last equation (θ) will give the stress S_a in the bar, it being the only unknown quantity in the equation. For R_1, R_2, R_3, &c., are found graphically, or can be calculated statically, they being the stresses in the members 1, 2, 3, &c., due to the external forces alone. The coefficients $c_1{}^a$, $c_2{}^a$, &c., can also be found from the dimensions of the structure; and m_1, m_2, &c., the coefficients of extension, are also known when the dimensions of each member are given. For only one superfluous member equation (θ) becomes

$$\begin{aligned} O &= S_a\, m_a + c_1{}^a\, m_1\, (R_1 + c_1{}^a\, S_a) + c_2{}^a\, m_2\, (R_2 + c_2{}^a\, S_a) \\ &\qquad + c_3{}^a\, m_3\, (R_3 + c_3{}^a\, S_a) + \&c. \\ &= S_a [m_a + m_1\, (c_1{}^a)^2 + m_2\, (c_2{}^a)^2 + m_3\, (c_3{}^a)^2 + \&c.] \\ &\quad + R_1\, m_1\, c_1{}^a + R_2\, m_2\, c_2{}^a + R_3\, m_3\, c_3{}^a + \&c. \end{aligned}$$

Should it be required to find the change in the length X Y under the influence of the external forces alone, without any superfluous members at all, we simply replace the redundant member A between X and Y by a pair of external forces S^a, as in fig. 128. Then, as δl now takes place in the direction of the force, the work done by the force will be of opposite sign to that done in deforming the members of the structure. The quantity $\delta l_a = S_a\, m_a$ will now be negative, and the last equation will become—

$$\begin{aligned} \delta\, l_a = S_a\, m_a = c_1{}^a\, m_1\, (R_1 + c_1{}^a\, S_a) + c_2{}^a\, m_2\, (R_2 + c_2{}^a\, S_a) \\ + c_3{}^a\, m_3\, (R_3 + c_3{}^a\, S_a) + \&c. \end{aligned}$$

As we want to find the deflection under the condition ot the external forces P, Q, &c., alone, S_a must be zero, and the above becomes—

$$\delta\, l_a = c_1{}^a\, m_1\, R_1 + c_2{}^a\, m_2\, R_2 + c_3{}^a\, m_3\, R_3 + \&c.$$

If there are redundant members in the structure, then the deflection will be obtained in the same way, but including those members.

As a numerical example of the foregoing, take the cantilever in the upper part of fig. 130. It is loaded at its lower extremity with $W_1 = 5$ tons; it is hinged at C, and tied back at D. The points C and D are maintained at a constant distance apart, which is equivalent to saying that the member 9 is absolutely rigid, and therefore its extension is zero. It is required to find the stress in each member of

the structure. The first thing to be done is to discover if there are any redundant members in the structure, and, if so, where they are situated. See what members may be discarded without destroying the structure. The members a, 2, 6, and 5 may be so removed, and we have left the structure shown in the lower part of the figure. The members 5, 6, and 2 can only be strained when a is strained, and there appears to be four redundant members. The necessary members are given by the equation *

$$m = 2j - 3.$$

Here the number of joints is six; hence $m = 9$; and as there are ten members in the structure, there can be only one redundant member. As 5, 6, and 2 are of no use without a, then a must be the redundant member. It will be

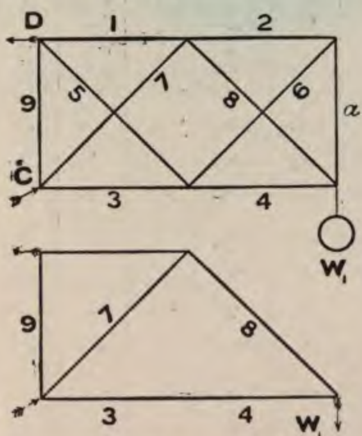

FIG. 130.

noticed that if 8, 7, and 4 were removed, the structure would not collapse; in fact, the structure is made up of two similar structures, one containing the members 1, 8, 4, 3, 7, and 9, fig. 131, while the other contains the members 1, 2, 6, 5, 3, and 9, the load being applied at the point where 2 meets 6. These two structures are superposed one on the other, and connected by the member a. Following out the method previously discussed, we remove the superfluous

* See Appendix.

bar a, and replace it by two forces equal to the stress in it acting at its extremities. Call these forces S_a. It will be most convenient to record in tabular form the results of the numerous calculations made.

TABLE REFERRING TO FIGURES 130 AND 131.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.
Member.	Length in inches.	Sectional area in square inches.	$m = \frac{l}{AE}$	R = stress in member due to external forces alone.	K^a = stress in member due to one ton applied at M.	k^a = stress in member due to one ton applied at N.	$c^a = K^a + k^a$	$c^a m$ R.	$m\,(c^a)^2$.
1	40	6	$\frac{1}{1800}$	$2\,W_1$	1	-2	-1	$-\frac{W_1}{900}$	$\frac{1}{1800}$
2	40	3	$\frac{1}{900}$	0	1	0	1	0	$\frac{1}{900}$
3	40	6	$\frac{1}{1800}$	$-W_1$	-2	1	-1	$\frac{W_1}{1800}$	$\frac{1}{1800}$
4	40	3	$\frac{1}{900}$	$-W_1$	0	1	1	$-\frac{W_1}{900}$	$\frac{1}{900}$
5	56 5	3	$\frac{1}{640}$	0	$\sqrt{2}$	0	$\sqrt{2}$	0	$\frac{1}{320}$
6	56·5	5	$\frac{1}{1060}$	0	$-\sqrt{2}$	0	$-\sqrt{2}$	0	$\frac{1}{530}$
7	56·5	5	$\frac{1}{1060}$	$-W_1\sqrt{2}$	0	$\sqrt{2}$	$\sqrt{2}$	$-\frac{W_1}{530}$	$\frac{1}{530}$
8	56·5	3	$\frac{1}{640}$	$W_1\sqrt{2}$	0	$-\sqrt{2}$	$-\sqrt{2}$	$-\frac{W_1}{320}$	$\frac{1}{320}$
9	40	∞	0	0	1	0	1	0	0
a	40	2	$\frac{1}{600}$	0	–	–	–	–	–

In the accompanying table, the first column contains the numbers which denote the several members, the second the length of the members in inches, and the third the sectional area of the members. In the fourth column is given the values of m, and the fifth contains the stress produced in each member when the redundant member has been removed. The sixth column contains the stress in the

members produced by unit force applied at M, fig. 131, while the seventh column gives a similar value when unit force is applied at N. The eighth column contains the sum of the numbers in the two previous columns.

Using the quantities given in this table to substitute in the equation (θ), namely,

$$O = S_a m_a + c_1{}^a m_1 (R_1 + c_1{}^a S_a) + c_2{}^a m_2 (R_2 + c_2{}^a S_a) \\ + c_3{}^a m_3 (R_3 + c_3{}^a S_a) + + + c_n{}^a m_n (R_n + c_2{}^a S_a),$$

which reduces to

$$S_a = -\frac{c_1{}^a m_1 R_1 + c_2{}^a m_2 R_2 + c_3{}^a m_3 R_3 + + + c_n{}^a m_n R_n}{m_a + m_1 (c_1{}^a)^2 + m_2 (c_2{}^a)^2 + m_3 (c_3{}^a)^2 + + + m_n (c_n{}^a)^2}$$

we obtain

$$S_a = \cdot 445\, W_1 = \cdot 445 \times 5 = 2\cdot 22 \text{ tons.}$$

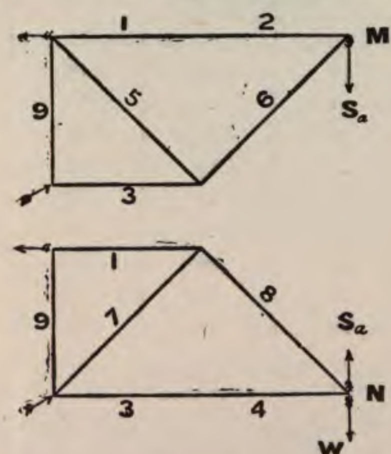

FIG. 131.

To find the stresses in the individual members, replace S_a, fig. 131, by 2·22 tons, and if we multiply all the numbers in Column VI. by 2·22, we shall obtain the stresses in the members due to 2·22 tons applied at M, fig. 131. These numbers are given in the second column of the next table.

The third column gives the stresses in members due to $W_1 - S_a$ applied at N in a downward vertical direction. This is given by the product of

$$(W_1 - S_a) \text{ and } \frac{R}{W_1}.$$

The true stress in any member is the sum of the numbers given in the second and third columns of the last table. The last column contains the stresses in members when the member a is assumed to support half the load W_1. It will be observed that the true stresses are not far from those,

Member.	$K_a S_a$ tons.	$(W_1 - S_a)\frac{R}{W_1}$ tons.	True stress in member in tons.	Stress in members with the assumption that half the load W_1 is supported by a.
1	2·2	5·6	7·8	7·5
2	2·2	..	2·2	2·5
3	− 4·4	− 2·8	− 7·2	− 7·5
4	..	− 2·8	− 2·8	− 2·5
5	3·1	..	3·1	3·5
6	− 3·1	..	− 3·1	− 3·5
7	..	− 3·95	− 3·95	− 3·5
8	..	3·95	3·95	3·5
9	2·2	..	2·2	2·5
a	..	..	2·2	2·5

obtained by the assumption mentioned above in this particular instance, but they may vary very considerably from those values if the transverse dimensions of the redundant member are much reduced.

It may be instructive to solve this same problem by the method of least work. Divide the structure, as shown in fig. 131, and let the upper portion be loaded at M with S_a tons, while the other portion is loaded at N with a force of W_1 tons downwards and S_a tons upwards, or a net force of $(W_1 - S_a)$ tons in the downward direction. The stress

produced by the upper load S_a alone in the different members will be found in the third column of the following table, and the stresses produced by the load $(W_1 - S_a)$ are given in the second column. The sums of the numbers in these two columns are given in the fourth column; while the work done by each stress is given in the last column in the form—

$$\frac{F^2 l}{2 A E} = \frac{F^2 m}{2}.$$

Member.	Stress in member produced by $(W_1 - S_a)$ alone at N.	Stress in member produced by S_a alone at M.	Total stress in member.	$m = \frac{l}{AE}$	Work done in straining member = (total stress)^2 $\times \frac{m}{2}$.
1	$2(W_1 - S_a)$	S_a	$2W_1 - S_a$	$\frac{1}{1800}$	$(2W_1 - S_a)^2 \times \frac{1}{3600}$
2	0	S_a	S_a	$\frac{1}{900}$	$S_a^2 \times \frac{1}{1800}$
3	$-(W_1 - S_a)$	$-2S_a$	$-(W_1 + S_a)$	$\frac{1}{1800}$	$(W_1 + S_a)^2 \times \frac{1}{3600}$
4	$-(W_1 - S_a)$	0	$-(W_1 - S_a)$	$\frac{1}{900}$	$(W_1 - S_a)^2 \times \frac{1}{1800}$
5	0	$S_a\sqrt{2}$	$S_a\sqrt{2}$	$\frac{1}{320}$	$S_a^2 \times \frac{1}{640}$
6	0	$-S_a\sqrt{2}$	$-S_a\sqrt{2}$	$\frac{1}{1060}$	$S_a^2 \times \frac{1}{1060}$
7	$-(W_1 - S_a)\sqrt{2}$	0	$-(W_1 - S_a\sqrt{2}$	$\frac{1}{1060}$	$(W_1 - S_a)^2 \times \frac{1}{1060}$
8	$(W_1 - S_a)\sqrt{2}$	0	$(W_1 - S_a\sqrt{2}$	$\frac{1}{320}$	$(W_1 - S_a)^2 \times \frac{1}{640}$
9	0	S_a	S_a	0	0
a	0	S_a	S_a	$\frac{1}{600}$	$S_a^2 \times \frac{1}{1200}$

The sum of all the quantities in this last column is the work done in deforming the structure. If this is differentiated with respect to the stress S_a in the redundant member, the differential coefficient must be zero. Let U denote the total work done; then

$$U = \frac{(2W_1 - S_a)^2}{3600} + \frac{S_a^2}{1800} + \frac{(W_1 + S_a)^2}{3600} + \frac{(W_1 - S_a)^2}{1800}$$

$$+ \frac{S_a^2}{640} + \frac{S_a^2}{1060} + \frac{(W_1 - S_a)^2}{1060} + \frac{(W_1 - S_a)^2}{640} + \frac{S_a^2}{1200}.$$

Collecting the quantities, and differentiating, we get—

$$O = \frac{d\,U}{d\,S_a} = -\ {\cdot}000554\,(2\,W_1 - S_a) + {\cdot}00779\,S_a$$

$$+\ {\cdot}000554\,(W_1 + S_a) - {\cdot}00632\,(W_1 - S_a)$$

and $$S_a = {\cdot}45\,W_1 \text{ approximately,}$$

a result similar to that obtained by the other method. The slight difference between the two numbers, ·45 W_1 and ·445 W_1, is due to the fact that these calculations have been

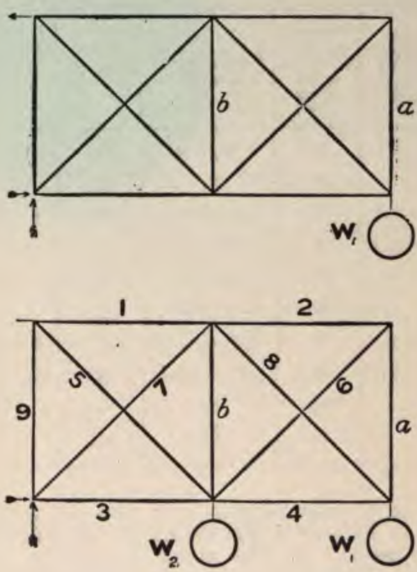

FIG. 132.

done with a pocket calculator, a form of circular slide rule, and numerically exact results have not been aimed at.

If in the cantilever in the last example a second superfluous member (*b*) is introduced, and at the same time the structure be loaded at two joints as shown in the lower part of fig. 132, the stresses in the members, including the redundant members, may be obtained in the manner previously

indicated in connection with equation (θ). Let the second load be denoted by W_2, and the stress in the bar b by S_b; also let the sectional area of this member be 4 square inches.

Remove the superfluous members, and replace the member (a) by unit stress acting at its extremities on the structure in the same way that S_a would do. This unit stress produces stresses in the several members represented by the symbol c^a with the suffix number of the member; *e.g.*, the stress so produced in (5) is $c_5{}^a$, the suffix a denoting that the unit stress is applied in the same manner as S_a. The different values of c^a are given in the sixth column of the following table. Now remove this unit stress and apply it in the position of the member b, and in the same manner that S_b would act. The stresses c^b produced by it in each member are found in the seventh column of the table. The fifth column contains the stresses produced by the external loads W_1 and W_2 when the superfluous bars a and b are removed.

TABLE IN CONNECTION WITH FIG. 132.

Member.	Length in inches.	Sectional area.	$m = \frac{l}{AE}$	R	c^a	c^b	$c^a\,m\,R$	$c^b\,m\,R$	$m\,(c^a)^2$	$m\,(c^b)^2$
1	40	6	$\frac{1}{1800}$	$2\,W_1$	-1	1	$-\frac{W_1}{900}$	$\frac{W_1}{900}$	$\frac{1}{1800}$	$\frac{1}{1800}$
2	40	3	$\frac{1}{900}$	0	1	0	0	0	$\frac{1}{900}$	0
3	40	6	$\frac{1}{1800}$	$-(W_1+W_2)$	-1	1	$\frac{W_1+W_2}{1800}$	$\frac{-(W_1+W_2)}{1800}$	$\frac{1}{1800}$	$\frac{1}{1800}$
4	40	3	$\frac{1}{900}$	$-W_1$	1	0	$-\frac{W_1}{900}$	0	$\frac{1}{900}$	0
5	56·5	3	$\frac{1}{640}$	$W_2\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$\frac{W_2}{320}$	$-\frac{W_2}{320}$	$\frac{1}{320}$	$\frac{1}{320}$
6	56·5	5	$\frac{1}{1060}$	0	$-\sqrt{2}$	0	0	0	$\frac{1}{530}$	0
7	56·5	5	$\frac{1}{1060}$	$-\sqrt{2}(W_1+W_2)$	$\sqrt{2}$	$-\sqrt{2}$	$\frac{-(W_1+W_2)}{530}$	$\frac{W_1+W_2}{530}$	$\frac{1}{530}$	$\frac{1}{530}$
8	56·5	3	$\frac{1}{640}$	$\sqrt{2}\,W_1$	$-\sqrt{2}$	0	$-\frac{W_1}{320}$	0	$\frac{1}{320}$	0
9	40	∞	0	$-W_2$	1	1	0	0	0	0
a	40	2	$\frac{1}{600}$	0	..	..	..	..	..	..
b	40	4	$\frac{1}{1200}$	0	..	..	..	..	..	..

Insert the several values found in the table, in the following equations, which are simply equations (θ) repeated, with the terms that are not required left out:—

$$O = S_a m_a + c_1^a m_1 [R_1 + c_1^a S_a + c_1^b S_b] + c_2^a m_2 [R_2 + c_2^a S_a + c_2^b S_b]$$
$$+ c_3^a m_3 [R_3 + c_B^a S_a + c_3^b S_b] + + + c_9^a m_9 [R_9 + c_9^a S_a + c_9^b S_b]$$
$$O = S_b m_b + c_1^b m_1 [R_1 + c_1^a S_a + c_1^b S_b] + c_2^b m_2 [R_2 + c_2^a S_a + c_2^b S_b]$$
$$+ c_3^b m_3 [R_3 + c_3^a S_a + c_3^b S_b] + + + c_9^b m_9 [R_9 + c_9^a S_a + c_9^b S_b]$$

and we obtain the following simultaneous equation—

$$\cdot 01503\, S_a - \cdot 0051\, S_b = \cdot 00664\, W_1 - \cdot 00177\, W_2$$
$$- \cdot 0072\, S_a + \cdot 0069\, S_b = - \cdot 003\, W_1 + \cdot 00612\, W_2,$$

which gives $S_a = \cdot 45\, W_1 + \cdot 28\, W_2,$
and $S_b = \cdot 035\, W_1 + 1 \cdot 18\, W_2.$

If the load W_2 were removed, the second superfluous bar (b) would have little effect in altering the stresses in the members; but if the load W_2 were replaced and W_1 removed, it then exerts considerable influence on the remainder of the structure. The load W_1 does not appreciably affect the stress in (b), but the load W_2 affects both a and b to a much greater extent.

Lattice Girder with One Superfluous Member.

In the upper part of fig. 133 is shown a short lattice girder, loaded at the joints of the lower boom and supported at its extremities. A much longer girder, or one having a greater number of diagonals, is generally to be found in actual practice; but the principle of solution is identical in both cases. The girder has been loaded at every joint to show the influence of every weight on the stresses in the members. The structure contains one more member than is required for statical equilibrium. This is shown in the equation $m = 2j - 3$.

Either of the end posts may be removed without the collapse of the structure. Let us remove number 17; it will then be found that numbers 5 and 13 are then useless, and we get the figure shown in the middle of fig. 133. Its appearance is peculiar, but if $W_1 = W_2 = W_3$, the several

stresses are given in the stress diagram on the right in the usual manner. But as it is more interesting to study the effect of every individual load on the structure, the stresses have been calculated by the "method of sections," and where necessary the ordinary polygon of forces. These

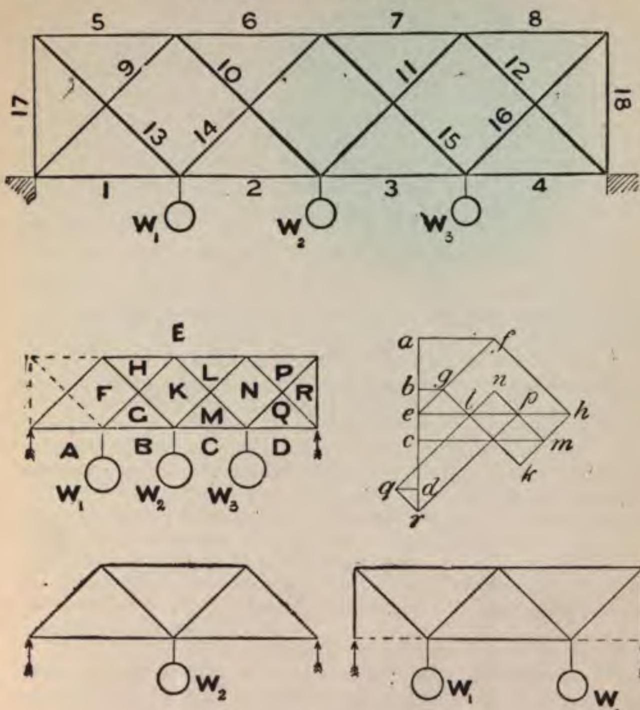

FIG. 133.

stresses are to be found in the fifth column of the following table. The members 5 and 13 are now replaced, and the stress S_a is inserted in the place of the redundant bar 17. (This stress is put in, in the positive direction, and if the numerical value is negative, it shows the stress is the opposite to tension, namely, compression.) The stress in

each member produced by unit stress applied in the place of S_a is given in the sixth column of the following table. The

TABLE OF QUANTITIES IN CONNECTION WITH FIG. 133.

Member.	Length in feet.	Sectional area, square inches.	$m = \frac{l}{AE}$.	R.	c^a	$m c^a R \times 10^6$.	$m (c^a)^2$.
1	6	6	001	$\frac{1}{4}(3 W_1 + 2 W_2 + W_3)$	1	$250 (3 W_1 + 2 W_2 + W_3)$	·001
2	6	12	·0005	$\frac{1}{4}(-W_1 + 2 W_2 + 3 W_3)$	-1	$125 (+W_1 - 2 W_2 - 3 W_3)$	·0005
3	6	12	·0005	$\frac{1}{4}(5 W_1 + 2 W_2 + 3 W_3)$	1	$125 (5 W_1 + 2 W_2 + 3 W_3)$	·0005
4	6	6	001	$-\frac{1}{4}(3 W_1 - 2 W_2 + W_3)$	-1	$250 (3 W_1 - 2 W_2 + W_3)$	·001
5	6	8	·00075	....	1	....	·00075
6	6	16	·000375	$-\frac{1}{2}(3 W_1 + 2 W_2 + W_3)$	-1	$187 (3 W_1 + 2 W_2 + 2 W_3)$	·000375
7	6	16	·000375	$-\frac{1}{4}(2 W_1 + 4 W_2 + 2 W_3)$	1	$-94 (2 W_1 + 4 W_2 + 2 W_3)$	·000375
8	6	8	·00075	$-(W_1 + W_3)$	-1	$750 (W_1 + W_3)$	·00075
9	8·5	17	·0005	$-\frac{1}{2\sqrt{2}}(3W_1 + 2W_2 + W_3)$	$-\sqrt{2}$	$250 (3 W_1 + 2 W_2 + W_3)$	·001
10	8·5	6	·0014	$\frac{1}{2\sqrt{2}}(3W_1 + 2W_2 + W_3)$	$\sqrt{2}$	$700 (3 W_1 + 2 W_2 + W_3)$	·0028
11	8·5	6	0014	$-\frac{1}{2\sqrt{2}}(3W_1 + 2W_2 + W_3)$	$-\sqrt{2}$	$700 (3 W_1 + 2 W_2 + W_3)$	·0028
12	8·5	17	·0005	$\frac{1}{2\sqrt{2}}(3W_1 + 2W_2 + W_3)$	$\sqrt{2}$	$250 (3 W_1 + 2 W_2 + W_3)$	·001
13	8·5	17	·0005	....	$-\sqrt{2}$	....	·001
14	8·5	12	·0007	$\sqrt{2} W_1$	$\sqrt{2}$	$1400 W_1$	·0014
15	8·5	12	·0007	$-\sqrt{2} W_1$	$-\sqrt{2}$	$1400 W_1$	·0014
16	8·5	17	·0005	$\sqrt{2} (W_1 + W_3)$	$\sqrt{2}$	$1000 (W_1 + W_3)$	·001
17	6	12	·0005	....	..	....	..
18	6	12	·0005	$-(W_1 + W_3)$	-1	$500 (W_1 + W_3)$	·0005

seventh column is the product of the numbers in the fourth, fifth, and sixth, multiplied by one million for the sake of getting rid of the cyphers after the decimal place. Inserting

the several quantities in equation (θ), and not forgetting to divide the sum of the quantities by one million, we get—

$$O = \cdot0005\,S_a + \cdot01347\,W_1 + \cdot0038\,W_2 + \cdot00465\,W_3 + \cdot01385\,S_a.$$

which reduces to

$$S_a = -\cdot94\,W_1 - \cdot265\,W_2 - \cdot325\,W_3\,;$$

the negative signs showing that each load helps to put the redundant member in compression.

For the sake of comparing the results obtained by assuming the girder to be composed of two simpler girders, and those obtained by the more correct method, a further table of comparative stresses has been made, and is given on page 291. The second column contains the stress in each member produced by the stress in the redundant member, namely—

$$S_a = -\cdot94\,W_1 - \cdot265\,W_2 - \cdot325\,W_3.$$

This is obtained by multiplying the stress S_a by the coefficient c^a in the sixth column of the previous table. The third column gives the actual stress in each member, and is obtained by adding together the quantities in the second and third columns. The last column contains the stress in each member when the girder is broken up into two elementary girders, as shown in the lower part of fig. 133. A comparison of the stresses thus obtained with the actual stresses will show at once how very wide of the mark they are. This column has been put in because it is so often stated in text-books that the girder can be broken up in this way, whereas the stresses thus determined do not at at all agree with the actual stresses.

In general, the bracing members are more numerous than are shown in fig. 133, but the method of calculation is the same in all respects.

A similar girder to that just discussed, but in which the inclination of the bracing members to the horizon is 60 deg. instead of 45 deg., is shown at A, fig. 134. It is used as a riveted structure for short spans, and as a pin structure for long spans. The Memphis and Indiana Bridges in the United States are of this form. Short-span riveted girders of this type are sometimes made with quadruple bracing, as shown at B. In general, the diagonals are riveted together where they cross one another, making the structure very rigid, similar to a plate girder. At C is shown a modification of the form shown in fig. 133, which is always used for long spans. The Pegram truss at D, fig. 134, is

an economical form, on account of the shortness of the compression members in the upper boom. There are three redundant members, two of which are shown dotted. A swing bridge is shown at E, containing three superfluous members. When sustaining a load, it rests upon four supports, and does not act as a cantilever, but as two girders supported at their ends. The Baltimore truss is shown at F, fig. 135, the dotted lines representing superfluous

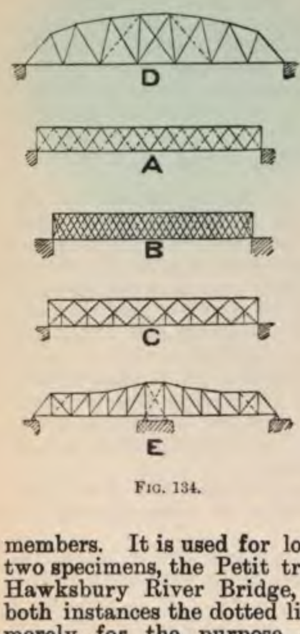

FIG. 134.

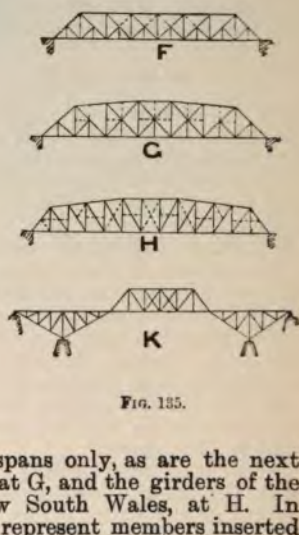

FIG. 135.

members. It is used for long spans only, as are the next two specimens, the Petit truss at G, and the girders of the Hawksbury River Bridge, New South Wales, at H. In both instances the dotted lines represent members inserted merely for the purpose of stiffening the compression members. A double cantilever with central girder is shown at K. The latter contains two superfluous members, and the outer ends of the cantilevers are anchored to the masonry. A very complete classification of bridges will be found in Professor Claxton Fidler's treatise on "Bridge Construction."

TABLE OF COMPARATIVE STRESSES.

Member	Stress in each member produced by redundant bar 17 $= c^a \times S_a$	True stress in each member $= R +$ stress produced by the redundant bar.	Stress in each member computed by dividing the girder up into two simple girders.
1	$-\cdot94W_1 - \cdot265W_2 - \cdot325W_3$	$-\cdot21W_1 + \cdot235W_2 + \cdot675W_3$	$\cdot5W_2$
2	$\cdot94W_1 + \cdot265W_2 + \cdot325W_3$	$\cdot69W_1 + \cdot765W_2 + 1\cdot075W_3$	$\cdot5W_1 + \cdot5W_2 + 5W_3$
3	$-\cdot94W_1 - \cdot265W_2 - \cdot325W_3$	$\cdot31W_1 + \cdot235W_2 + \cdot425W_3$	$\cdot5W_1 + \cdot5W_2 + \cdot5W_3$
4	$\cdot94W_1 + \cdot265W_2 + 325W_3$	$\cdot19W_1 + \cdot765W_2 + \cdot075W_3$	$5W_2$
5	$-\cdot94W_1 - \cdot265W_2 - \cdot325W_3$	$-\cdot94W_1 - \cdot265W_2 - \cdot325W_3$	$-\cdot75W_1 \quad -\cdot25W_3$
6	$\cdot94W_1 + \cdot265W_2 + \cdot325W_3$	$-\cdot56W_1 - \cdot735W_2 - \cdot175W_3$	$-\cdot75W_1 - \sqrt{2}W_2 - \cdot25W_3$
7	$-\cdot94W_1 - \cdot265W_2 - \cdot325W_3$	$-1\cdot44W_1 - 1\cdot265W_2 - \cdot825W_3$	$-\cdot25W_1 - \sqrt{2}W_2 - \cdot75W_3$
8	$\cdot94W_1 + \cdot265W_2 + \cdot325W_3$	$-\cdot06W_1 + \cdot265W_2 - \cdot675W_3$	$-\cdot25W_1 \quad -\cdot75W_3$
9	$1\cdot32W_1 + \cdot375W_2 + \cdot52W_3$	$\cdot27W_1 - \cdot325W_2 + \cdot17W_3$	$-\cdot7W_2$
10	$-1\cdot32W_1 - \cdot375W_2 - \cdot52W_3$	$-\cdot27W_1 + \cdot325W_2 - \cdot17W_3$	$\cdot7W_2$
11	$1\cdot32W_1 + \cdot375W_2 + \cdot52W_3$	$\cdot27W_1 - \cdot325W_2 + \cdot17W_3$	$\cdot7W_2$
12	$-1\cdot32W_1 - \cdot375W_2 - \cdot52W_3$	$-\cdot27W_1 + \cdot325W_2 - \cdot17W_3$	$-\cdot7W_2$
13	$1\cdot32W_1 + \cdot375W_2 + \cdot52W_3$	$1\cdot32W_1 + \cdot375W_2 + \cdot52W_3$	$1\cdot06W_1 \quad + \cdot35W_3$
14	$-1\cdot32W_1 - \cdot375W_2 - \cdot52W_3$	$\cdot094W_1 - \cdot375W_2 - \cdot52W_3$	$\cdot35W_1 \quad - \cdot35W_3$
15	$1\cdot32W_1 + \cdot375W_2 + \cdot52W_3$	$-2\cdot734W_1 - \cdot375W_2 - \cdot52W_3$	$-\cdot35W_1 \quad + \cdot35W_3$
16	$-1\cdot32W_1 - \cdot375W_2 - \cdot52W_3$	$\cdot094W_1 - \cdot375W_2 + \cdot894W_3$	$\cdot35W_1 \quad + 1\cdot06W_3$
17	$-\cdot94W_1 - \cdot265W_2 - \cdot325W_3$	$-\cdot94W_1 - \cdot265W_2 - \cdot325W_3$	$-\cdot75W_1 \quad - \cdot25W_3$
18	$\cdot94W_1 + \cdot265W_2 + \cdot325W_3$	$-\cdot06W_1 + \cdot265W_2 - \cdot675W_3$	$-\cdot25W_1 \quad - \cdot75W_3$

DEFLECTION.

Given any structure composed of members hinged together at their extremities, it is required to find the deflection of any point in that structure, in any direction, due to any load; and further, to find the force that must be applied at that point to *prevent* any deflection of that point. A very simple pin structure has been selected, fig. 136, the left-hand end being *fixed*, and the right-hand end free to move horizontally on a frictionless support. The three upper joints are loaded with the weights indicated in the figure in tons, by

numbers attached to the arrows. The span is 40 ft., and the lengths of the individual members are to be found in the accompanying table, in the second column.

TABLE OF QUANTITIES RELATING TO FIG. 136.

Member.	Length in feet.	Sectional area in square inches.	Stress in tons.	$1000\,m = 1000\,\frac{l}{AE}$	c^a	$R\,c^a\,m$	$1000 \times (c^a)^2 m$	c^b	$R\,c^b\,m$	$1000 \times (c^b)^2 m$	$1000 \times m\,c^a\,c^b$
1	21·5	14	84	1·52	−2·33	−·3	8·6	−1·1	−·14	1·84	3·9
2	21·5	14	84	1·52	−2·33	−·3	8·6	−1·1	−·14	1·84	3·9
3	21·5	14	84	1·52	−2·33	−·3	8·6	−1·1	−·14	1·84	3·9
4	21·5	14	84	1·52	−2·33	−·3	8·6	−1·1	−·14	1·84	3·9
5	7·5	4	24	1·86	− ·75	−·034	1·05	−·37	−·0165	·25	·52
6	10	3·5	20	2·9	− ·75	−·043	1·62	−·1	−·0058	·03	·28
7	7·5	4	24	1·86	− ·75	−·034	1·05	−·37	−·0165	·25	52
8	23·5	2	11	11·75	0	0	0	−·85	−·11	1	0
9	23·5	2	11	11·75	0	0	0	−·85	−·11	1	0
10	24	30	−104	·8	1·52	−·127	1·86	1·45	−·12	1·68	1·77
11	24	30	−97	·8	1·35	−·105	1·47	2	−·155	3·2	2·16
12	24	30	−97	·8	1·35	−·105	1·47	2	−·155	3·2	2·16
13	24	30	−104	·8	1·52	−·127	1·86	1·45	−·12	1·68	1·77
..	..	..	..	..	Sum	−1·775	·04478	..	−1·3688	·0196	·02478

The sectional area of each member is given in the third column of the table, and the stress in each member, as found by the stress diagram, is given in the tourth column.

We will first proceed to find the deflection horizontally of the right-hand end. If a superfluous member be inserted, tying the two ends together, the stress in which is S_a, we shall have the relation already indicated in equation (θ), namely,

$$O = S_a m_a + c_1{}^a m_1 (R_1 + c_1{}^a S_a) + c_2{}^a m_2 (R_2 + c_2{}^a S_a) \\ + c_3{}^a m_3 (R_3 + c_3{}^a S_a) + \&c.,$$

in which $S_a m_a$ is the extension δl_a of the member (a), as shown by the equation (ζ). If the movement of the end of structure take place when there is a member (a) inserted, it will be of the same sign as the other extensions, because they all take place in directions opposite to those in which the stresses act. If the superfluous member (a) be replaced by forces S_a equal to the stress in that member, the sign of

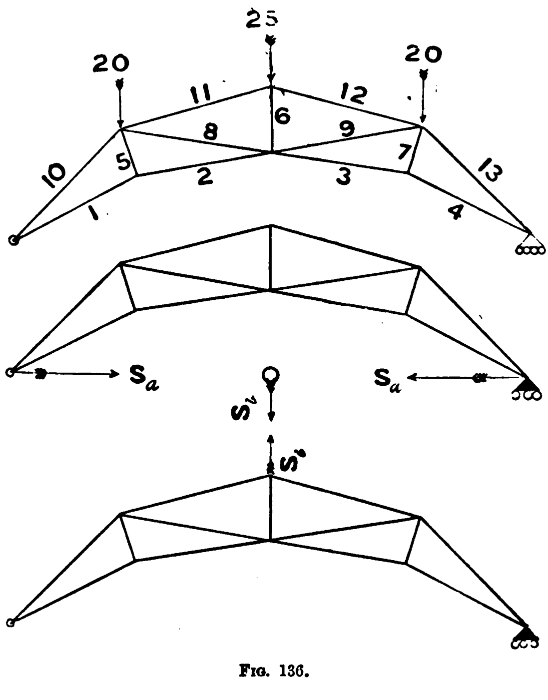

FIG. 136.

the movement of the end under the action of S_a will be of opposite sign to the other extensions, because it takes place in the same direction as the force acts. This was explained in connection with equation (δ). Replace the member (a) by a pair of forces S_a, as indicated in the middle of the fig. 136, then the deflection $S_a m_a = \delta l_a$ is of opposite sign tc

that of all the other terms in the above equation ; consequently we may write—

$$S_a m_a = \delta l_a = c_1{}^a m_1 (R_1 + c_1{}^a S_a) + c_2{}^a m_2 (R_2 + c_2{}^a S_a) + c_3{}^a m_3 (R_3 + c_3{}^a S_a) + \&c.,$$

in which δl_a is the deflection of the free end.

The member (a) does not exist in the upper part of the figure ; therefore $S_a = O$, and

$$\delta l_a = m_1 R_1 c_1{}^a + m_2 R_2 c_2{}^a + m_3 R_3 + c_3{}^a + \&c.,$$

= sum of the quantities in the seventh column of the table,

$= - 1{\cdot}775$ in.

The result is in inches, because all lengths have been reduced to inches. The negative sign indicates that the movement takes place in a direction opposite to the supposed force S_a.

If the free end is prevented from moving under the load, δl_a will be zero, and S_a will be the force which must be applied to the end horizontally to prevent deflection. In the last equation but one, put $\delta l_a = O$, and we have

$$O = - 1{\cdot}77 + S_a[(c_1{}^a)^2 + (c_2{}^a)^2 + (c_3{}^a)^2 + \&c.],$$
$$= - 1{\cdot}77 + {\cdot}0448\, S_a,$$

and $S_a = 39{\cdot}5$ tons.

Now tie the two ends together with a superfluous member (a), whose sectional area is 5 square inches. Using the last equation but three, which is a repetition of equation (θ), we get—

$$O = {\cdot}008\, S_a - 1{\cdot}77 + {\cdot}0448\, S_a,$$

from which $S_a = 33{\cdot}5$ tons.

Also the extension δl_a of the member (a)

$$= \text{strain} \times \text{length} = \frac{\text{stress}}{E} \times l$$

$$= - \frac{6{\cdot}7 \times 40 \times 12}{12000} = - {\cdot}268 \text{ in.}$$

Hence the deflection of the free end horizontally when the redundant member is put in is $- {\cdot}268$ in.

Now take out the superfluous member (a), and find the vertical deflection of the topmost point of the structure.

Proceeding as before, we put in a pair of forces in the line of the required deflection, these forces really being substituted for the stress in a redundant member occupying the same position. Let suffix b denote the quantities referring to this second superfluous member or its equivalent forces. Using the second of the equations (θ), we have, after changing $S_b m_b$ into $-\delta l_b$,

$$\delta l_b = c_1{}^b m_1 R_1 + c_2{}^b m_2 R_2 + c_3{}^b m_3 R_3 + \&c.$$

The right-hand side of this equation will be found in the tenth column of the previous table, and

$$\delta l_b = -1{\cdot}37 \text{ in.},$$

the negative sign showing that the deflection took place in a direction opposite to that in which the force $S_b (= O)$ was applied. (See lower part of fig. 136.)

If the superfluous member (a) exists in the structure, the deflection of the topmost point will be very different from that found above. Thus, from equation (θ),

$$\begin{aligned}\delta l_b &= c_1{}^b m_1 (R_1 + c_1{}^a S_a) + c_2{}^b m_2 (R_2 + c_2{}^a S_a) \\ &\qquad + c_3{}^b m_3 (R_3 + c^a S_a) + \&c \\ &= \Sigma (R m c^b) + S_a(c_1{}^a c_1{}^b m_1 + c_2{}^b c_2{}^a m_2 + c_3{}^b c_3{}^a m_3 + \&c. \\ &= -1{\cdot}37 + 33{\cdot}5 \times {\cdot}02478 \\ &= -{\cdot}54 \text{ in.}\end{aligned}$$

The quantities inside the second bracket above are given in the last column of the foregoing table.

Lastly, let there be a redundant member (b) (sectional area 6 square inches and 10 ft. long) as well as (a) existing in the structure at the same time; then from equation (θ) we get the simultaneous equation—

$$\begin{aligned}O &= S_a m_a + c_1{}^a m_1 (R_1 + c_1{}^a S_a + c_1{}^b S_b) + c_2{}^a m_2 \\ &\quad (R_2 + c_2{}^a S_a + c_2{}^b S_b) + c_3{}^a m_3 (R_3 + c_3{}^a S_a + c_3{}^b S_b) \times \&c. \\ O &= S_b m_b + c_1{}^b m_1 (R_1 + c_1{}^a S_a + c_2{}^b S_b) + c_2{}^b m_2 \\ &\quad (R_2 + c_2{}^a S_a + c_2{}^b S_b) + c_3{}^b m_3 (R_3 + c_3{}^a S_a + c_3{}^b S_b) + \&c.;\end{aligned}$$

and after substituting and reducing, there remains

$$1{\cdot}77 = {\cdot}0528\, S_a + {\cdot}02478\, S_b,$$
$$1{\cdot}37 = {\cdot}02478\, S_a + {\cdot}0017\, S_b,$$

from which we obtain

$$S_a = 7{\cdot}05 \text{ tons and } S_b = 56{\cdot}7 \text{ tons};$$

also $$\delta l_a = -\frac{7 \cdot 05}{5} \times \frac{40 \times 12}{12000} = -\cdot 056 \text{ in.},$$

and $$\delta l_b = -\frac{56 \cdot 7}{6} \times \frac{10 \times 12}{12000} = -\cdot 095 \text{ in.}$$

Trussed Pine Beam.

Thus far we have dealt only with hinged structures having superfluous members, but it is a common occurrence to find structures composed partly of hinged members and partly of continuous elastic members. The method of treatment of such problems will be here indicated, but space will not permit of many examples being worked out.

The trussed beam, fig. 137, is 40 ft. long, and rectangular in section. It is loaded at any point D with 4 tons. It is required to find the tension in the two tie rods, the compressive stress in the strut, and the maximum stress, both tensile and compressive, in the beam. The beam is 15 in. deep and 12 in. wide, and is made of pine, whose modulus of elasticity is about 600 tons per square inch. The tie rods are of wrought iron 20·45 ft. long, and 2 square inches in sectional area. The strut is of cast iron 4 ft. long, 4 square inches in sectional area, and whose modulus of elasticity is about 7,000 tons per square inch. The point D is at quarter span.

Let P be the thrust in the strut in tons; then the work done on the strut is given by the expression

$$P^2 \frac{m}{2} = \cdot 000857 \; P^2 \text{ inch-tons.}$$

Similarly, if S represent the stress in each of the tie rods, then

$$S^2 \frac{m}{2}$$

is the work done on each of them. But the stresses in the tie rods and the stress P are represented by the three sides of a triangle, from which we find

$$S = 2 \cdot 55 \; P;$$

therefore the work done on the *two* ties

$$= 2 \times (2 \cdot 55 \; P)^2 \times \frac{m}{2} = \cdot 0671 \; P^2 \text{ inch-tons.}$$

The work done in compressing the beam by the horizontal components of the stresses in the tie rods

$$=\left(\frac{S}{5\cdot11}\right)^2 \times \frac{m}{2} = \frac{(2\cdot55\ P)^2}{26\cdot1} \times \frac{40 \times 12}{12 \times 15 \times 600} = \cdot00115\ P^2 \text{ inch-tons.}$$

There is, lastly, the work done by P in bending the beam.

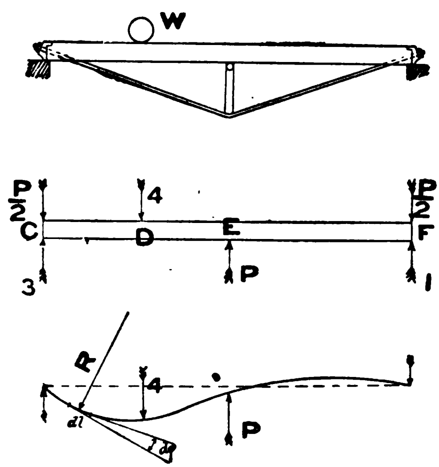

FIG. 137.

*It was shown [equation (γ)] that the work done by any force on a piece of material, when working in conjunction with other forces, was

$$\frac{\text{force in question}}{2} \times \text{sum of all the forces} \times m.$$

In the same way the work done in bending an element of material by a force in conjunction with other forces is

$$\frac{\text{bending moment}}{2} \times \text{angle through which element is bent.}$$

The bending moment is the average over the element due to the first-mentioned force alone; while the angle through which the element is bent is that due to all the forces acting on the element. The beam in fig. 137 is subject to the forces

* See page 270.

shown in the middle part of the figure, namely, at the left-hand end the supporting force upwards is 3 tons and the component of S in a vertical direction downwards, which equals $\cdot 5$ P; at E there will be the upward thrust of the strut, and at the right-hand end there will be the supporting force of 1 ton upwards and a downward force $\cdot 5$ P, equal to the vertical component of S.

Let R be the radius of curvature of the neutral surface of the beam at any point, and $d\phi$ the small angle between the tangents at the extremities of an element of the neutral line, fig. 137 (lower portion), whose length is dl. Then

$$\mathrm{R}\, d\phi = dl,$$

and $$d\phi = dl \cdot \frac{1}{\mathrm{R}} = dl \cdot \frac{\mathrm{M}}{\mathrm{EI}} \text{ [see equation (29)].}$$

This small increment of angle $d\phi$ is due to *all* the forces acting on the beam; similarly, M is the average moment over the element of beam producing the increment of angle $d\phi$. As we require to find the work done on the whole structure by the force P, the work done by P in bending the element of the beam will be $\frac{1}{2} \times$ moment over element caused by P, $\times\, d\phi$.

Let the moment over the element due to the force P alone be M_1, then work done by P in bending beam

$$= \int \frac{\mathrm{M}_1}{2}\, d\phi,$$

the integration to be taken in sections over the whole length of the beam. The reason for integrating in sections from F to E, from E to D, and from D to C, is that in passing over the end of any one of these sections the generating function is not continuous in the sense used in the calculus, and an expression can only be integrated between certain limits when the generating function is finite and continuous between those limits.

Now, inserting the value of $d\phi$ obtained above, we obtain for the work done

$$\int \frac{\mathrm{M}_1}{2} \times \frac{\mathrm{M}}{\mathrm{EI}} \cdot dl.$$

Between F and E the bending moment M, at any section G, distant l from F, is

$$\left(\frac{\mathrm{P}}{2} - \frac{\mathrm{W}}{4}\right) l \text{ ton-inches,}$$

and the bending moment M_1 due to P is

$$\frac{P}{2} l \text{ ton-inches.}$$

Inserting these values in the above equation, we have for the work done by P, in bending the beam between F and E,

$$\frac{1}{2EI}\int_o^{\frac{L}{2}} \frac{P}{2}\left(\frac{P}{2} - \frac{W}{4}\right) l^2\, dl,$$

where L denotes the length of span in inches.

Performing the integration, we get for the work between F and E,

$$\frac{P}{4EI}\left(\frac{P}{2} - \frac{W}{4}\right)\left[\frac{l^3}{3}\right]_o^{\frac{L}{2}} \text{ inch-tons}$$

$$= \frac{PL^3}{192EI}\left(P - \frac{W}{2}\right) \text{ inch-tons.}$$

Between C and D the bending moment due to P will be

$$\frac{P}{2} \times l$$

where l is measured from C to the right, and the bending moment due to all the forces will be

$$\left(\frac{P}{2} - \tfrac{3}{4} W\right) l,$$

and the work done between C and D will be

$$\frac{1}{2EI}\int_o^{\frac{L}{4}} \frac{Pl^2}{2}\left(\frac{P}{2} - \tfrac{3}{4} W\right) dl$$

$$= \frac{P}{8EI}(P - \tfrac{3}{2} W)\frac{L^3}{192} \text{ inch-tons.}$$

Between D and E the bending moment due to P will be

$$\frac{P}{2} l$$

when l is measured from C. The bending moment due to all the forces will be

$$\left(\frac{P}{2} - \tfrac{3}{4} W\right) l + W\left(l - \frac{L}{4}\right),$$

and the work done by P between D and E

$$= \frac{1}{2\,\mathrm{E\,I}} \int_{\frac{L}{4}}^{\frac{L}{2}} \frac{P}{8} [2\,P\,l^2 + W\,l^2 - W\,L\,l]\, d\,l$$

$$= \frac{P}{16\,\mathrm{E\,I}} (14\,P - 11\,W) \frac{L^3}{192} \text{ inch-tons.}$$

Collecting terms, we have for the work done by P in bending the beam from end to end—

$$\frac{P\,L^3}{192\,\mathrm{E\,I}} \left[P - \frac{W}{2} + \frac{P}{8} - \frac{3\,W}{16} + \frac{14\,P}{16} - \tfrac{11}{16}\,W\right]$$

$$= \frac{P\,L^3}{192\,\mathrm{E\,I}} (2\,P - \tfrac{11}{8}\,W)$$

$$= {\cdot}57\,P^2 - {\cdot}39\,P\,W \text{ inch-tons.}$$

Then the total work done by P on the whole structure, including itself, is zero; or,

$${\cdot}000857\,P^2 + {\cdot}0671\,P^2 + {\cdot}00115\,P^2 + {\cdot}57\,P^2 - {\cdot}39\,P\,W = 0,$$

and $P = {\cdot}61\,W.$

If W is 4 tons, then

$$P = 2{\cdot}44 \text{ tons;}$$

also, $S = 2{\cdot}55\,P = 6{\cdot}23 \text{ tons.}$

$$\text{Stress in tie rods} = \frac{6{\cdot}23}{2} = 3{\cdot}1 \text{ tons per square inch.}$$

$$\text{Stress in strut} = \frac{2{\cdot}44}{4} = {\cdot}61 \text{ tons per square inch.}$$

The maximum bending moment occurs at D, and equals 213·6 ton-inches, which equals $\frac{1}{6} f\,b\,d^2$, where f = maximum stress in material, b is the breadth, and d the depth. Solving for f, we find f = ·475 ton per square inch.

Lattice Girder with Riveted Bracing.

A lattice girder is really a structure with redundant members, but when the bracing is close together, and riveted where they cross one another, the girder is made so rigid that it approaches very near to the ordinary plate girder, and can be treated as such without introducing an error of

any great magnitude. It was shown in the design of a plate girder (figs. 99 to 102) that, first, the shear stress over the web was approximately constant; and second, that there were in the web induced tensile and compressive stresses of equal intensity, over planes at right angles to one another, and inclined at 45 deg. to the direction of the direct stresses in the flanges. Referring to fig. 138, the induced tensile stress in the web is shown along a strip at C, and the

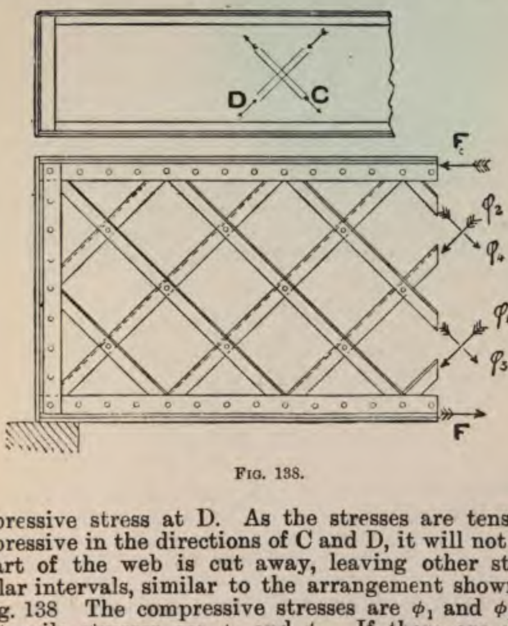

FIG. 138.

compressive stress at D. As the stresses are tensile and compressive in the directions of C and D, it will not matter if part of the web is cut away, leaving other strips at regular intervals, similar to the arrangement shown below in fig. 138 The compressive stresses are ϕ_1 and ϕ_2, while the tensile stresses are ϕ_3 and ϕ_4. If these are resolved vertically, it will be noticed their components all have the same sense of direction—*i.e.*, downwards. As these forces represent the action of the removed parts on the remaining parts, it is evident the removed parts will tend to move downwards over those that remain; in other words, the shearing force at the section in question tends to make the right-hand piece slide downwards over the left-hand piece. The sum of the vertical components equals the shearing

force at that section, or, if θ is the inclination of the bracing to the horizon,

$$\text{shearing force} = \phi_1 \sin\theta + \phi_2 \sin\theta + \phi_3 \sin\theta + \phi_4 \sin\theta.$$

As the shearing stress over any section in a plate girder is approximately constant, the stresses ϕ_1, ϕ_2, &c., will be approximately equal, if the dimensions of the braces are the same at that section. In this way the necessary bracing can be calculated for, if the loads, &c., are given; and conversely, if the dimensions and loads are given, the stresses may be calculated.

Three-hinged Arch.

Let C E D, fig. 139, represent the centre line of a three-hinged arch, the hinges being at C, E, and D. Let the clear span be l feet, and the rise h feet at centre of span. Also let a load w tons be situated as shown in the figure, at a distance of x feet hozizontally from the point E. The horizontal component of the supporting forces is H, and the vertical components are represented by V_1 and V_2. Taking moments about D, we get

$$V_1 l - w\left(\frac{l}{2} + x\right) = o,$$

or
$$V_1 = \frac{w}{l}\left(\frac{l}{2} + x\right);$$

also
$$V_2 = w - V_1 = \frac{w}{l}\left(\frac{l}{2} - x\right).$$

Considering the forces to the right of E, the bending moment at that point is

$$V_2 \times \frac{l}{2} - H h \text{ ton-feet},$$

and as there is a hinge at E, there can be no bending moment there; consequently the last expression must be equal to zero. This gives

$$H = \frac{w}{2h}\left(\frac{l}{2} - x\right)$$

The two parts of the arch are simply curved beams, and can be treated as such. The complete reaction of one beam on the other at E is obtained by combining the vertical shearing force at E with the horizontal component H.

In the present case it will be a force inclined upwards to the left. Only one load is given in the figure, but the method of procedure is the same for any number. The graphical method of treating the same problem is interesting. As before, neglecting the weight of the structure, the reaction at D (lower part of figure) must pass through E, as there are only two forces acting on the right-hand half, and for equilibrium these must be equal and opposite, and consequently must lie in the line D E. Produce D E to cut the line of action of w in F, and join C F. Then C F D is the funicular polygon for that particular load; and if the pole diagram be drawn by setting down the load line $b\,k = w$,

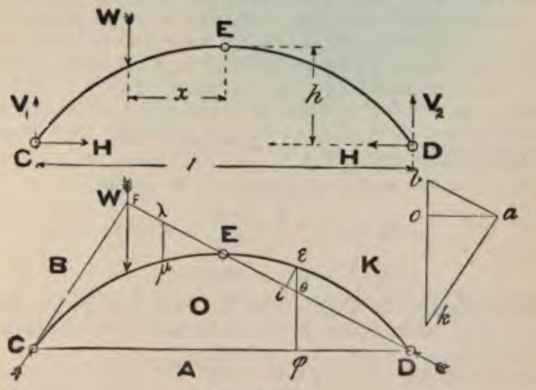

FIG. 139.

and through b and k respectively, drawing $b\,a$ and $k\,a$ parallel to F D and C F, and finally the horizontal a O, then, according to equation (20), the bending moment at any point ϕ in the line C D (treated as a single beam supported at C and D)

$$= \text{the intercept } \theta\phi \times o\,a \times \text{scale of } \theta\phi \times \text{scale of } o\,a.$$

In addition to the bending moment due to the vertical components, as in a beam, there is the bending moment due to the horizontal component of the reactions, and this will be of opposite sign to the previous moment, and will equal

$$\epsilon\phi \times \text{scale of } \epsilon\phi \times \text{H},$$

which equals

$$\epsilon\phi \times \text{scale of } \epsilon\phi \times o\,a \times \text{scale of } o\,a.$$

The total bending moment at ϵ in the arch is the sum of these two moments—that is,

$$(\epsilon\phi - \theta\phi) \times oa \times \text{scales} = \epsilon\theta \times oa \times \text{scales}.$$

Similarly, the bending moment at μ in the arch is

$$\lambda\mu \times oa \times \text{scales},$$

but this moment is of opposite sign to that at ϵ. because the intercept $\lambda\mu$ is outside the arch, while $\epsilon\theta$ is inside.

We may therefore conclude that, in general, the intercept between the funicular polygon and the centre line of the arch ring × the pole distance $o\,a$ × the scales, will always be the bending moment when the arch is hinged at its extremities, and when the funicular polygon passes through those extremities. This result may have been obtained otherwise, thus: The resultant force in the arch at ϵ is along D E, which is parallel to and represented by $b\,a$. Therefore, moment at $\epsilon = b\,a$ × scale of $b\,a \times \epsilon i$ × scale of ϵi. But the triangle $\epsilon i \theta$ is similar to the triangle $b\,o\,a$ therefore: the bending moment at ϵ

$$= b\,a \times \epsilon\theta \sin \epsilon\theta i \times \text{scales}$$

$$= b\,a \times \epsilon\theta \times \frac{o\,a}{b\,a} \times \text{scales}$$

$$= \epsilon\theta \times o\,a \times \text{scales}.$$

As there can be no moment at E, there can be no intercept there, and consequently the funicular polygon must pass through E.

Summing up, we may state that the bending moment at any point in an arch ring can be expressed as—the vertical intercept at that point between the funicular polygon and arch ring × horizontal pole distance × scales; or by the perpendicular let fall from the point in the arch ring on to the nearest side of the funicular polygon × the force in the pole diagram corresponding to that side in the funicular polygon × scales.

The same holds whether there are two or three hinges, provided that the funicular polygon be so drawn that it passes through the two hinges situated at the springers of the arch. An example is given in the upper part of fig. 140. In the same figure is represented a section through the above two-hinged arch at C, and which has a solid web. We find, adjacent to the section A B, the line F G of the funicular polygon, which means that the force R, acting along G F, maintains the remainder of the arch in equilibrium, together

with the forces to the left of the section A B. From C, the neutral point of A B, drop a perpendicular on to F G, and let its length be p; then the bending moment at A B is

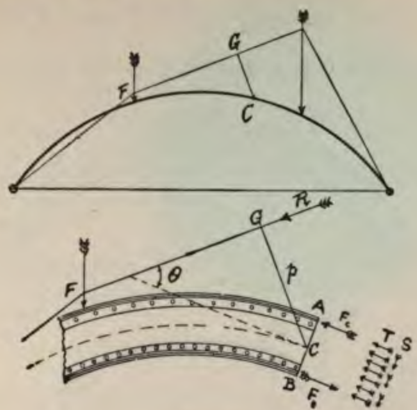

FIG. 140.

R × p. Resolve R along and perpendicular to the neutral fibre at C, and we have—

$$\text{R} \cos \theta = \text{direct thrust over section A B} = \text{T},$$
$$\text{R} \sin \theta = \text{shearing force over A B} = \text{S}.$$

ARCH WITH NO HINGES.

In fig. 141 is shown an arch rigidly attached to the foundations at the abutments, and lower down in the figure will be found the neutral line of the arch ring. Consider any element of the arch ring whose length is $d\,s$, and whose co-ordinates at its middle point are x and y. Also let the load on the portion of the arch to the left of $d\,s$ be W, acting through its centre of gravity, as shown in the figure, at a distance X from $d\,s$. Further, let the horizontal component of the reaction at A be H_1, and the vertical component be V_1. Let the moment at A be represented by M_1. Now, if the ends of the arch are rigidly maintained in position,

the total change in the angle between the sections A and B will be zero—*i.e.*, according to equation on page 298.

$$\int_A^B \frac{M}{EI} \cdot ds = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\kappa)$$

the integration being performed over the arch between A and B.

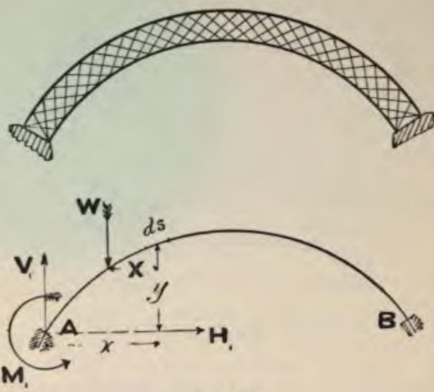

FIG. 141.

In the above expression to be integrated E is constant, and if I is not constant, its value in terms of x and y, or s, must be known and inserted in the expression before integration. The value of M is the sum of the moments acting on the left of the element $d\,s$, which equals

$$M_1 + H_1\,y - V_1\,x + w\,X.$$

The value of $d\,s$ in terms of x and y (or *vice versâ*) must be inserted also before integrating.

It is often convenient to perform the integration by cutting up the arch into a number of segments, and calculating the average value of

$$\frac{M}{EI}\,d\,s$$

for each segment, and then adding them all together. The result is exceedingly near the accurate integration.

As the points of support do not move under any circumstances, the deflection of either point horizontally or vertically is zero, and the work done by H_1 or V_1 on the structure is zero.

Let $d\,h$ be the horizontal displacement of either end. Work done by H_1

$$= H_1 \frac{d\,h}{2}, \quad \dots\dots\dots \quad (\pi)$$

and this must equal the work done by H_1 in deforming the structure.

The change of angle between the ends of the element $d\,s$ is

$$\frac{M}{E\,I} \cdot d\,s,$$

and the work done by H_1 in bending element $d\,s$ through this angle $= \frac{1}{2} \times$ moment over element produced by $H_1 \times$ change of angle

$$= \tfrac{1}{2} \times H_1\,y \times \frac{M}{EI}\,d\,s,$$

and total work done by H_1 in bending arch ring

$$= \int_A^B \frac{H_1\,y\,M}{2\,E\,I} \cdot d\,s = \frac{H_1}{2\,E} \int_A^B \frac{M\,y}{I} \cdot d\,s. \quad \dots \quad (\rho)$$

Let A be the average sectional area of element $d\,s$, and T the total thrust along the axis of element, while ϕ is the angle made by the axis of the element with the axis of x; then

$$T = V_1 \sin\phi + H_1 \cos\phi - W_1 \sin\phi.$$

Extension or compression of element

$$= \frac{T \,.\, d\,s}{A\,E},$$

and work done by H_1 in compressing $d\,s$

$$= \frac{H_1 \cos\phi}{2} \times \frac{T \,.\, d\,s}{A\,E}.$$

Also total work done by H_1 in compressing the whole arch ring

$$= \frac{H_1}{2\,E} \int_A^B \frac{T \cos\phi}{A} \cdot d\,s. \quad \dots\dots \quad (6)$$

Over the element $d\,s$ there will be a shearing force S, say; then the work done by H_1 in tending to shear the element transversely

$$= \frac{H_1 \sin\phi}{2} \times \frac{S \,.\, d\,s}{A\,G}$$

where G is the transverse modulus of elasticity, sometimes called the shear modulus.

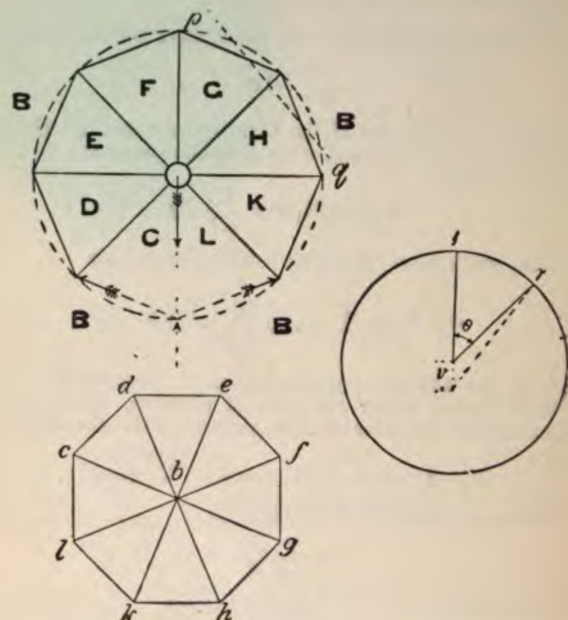

FIG. 142.

The total work done over the whole arch rib by H_1 in tending to shear

$$= \frac{H_1}{2\,G} \int_A^B \frac{S \sin\phi}{A} . \, d\,s. \quad \ldots \ldots \quad (r)$$

Collecting the above quantities together, we obtain for the total work done by H_1 on the structure—

O = work done by H_1 in displacing ends of arch rib
+ work done by H_1 in altering the angle between the ends (bending)
+ work done by H_1 in compressing the arch rib;
+ work done by H_1 in shearing the arch rib.

The first of these quantities is zero, because the deflection is zero. We then have left—

$$O = \frac{H_1}{2E}\int_A^B \frac{M\,y}{I}\,.\,d\,s + \frac{H_1}{2E}\int_A^B \frac{T\cos\phi}{A}\,.\,d\,s$$

$$+ \frac{H_1}{2G}\int_A^B \frac{S\sin\phi}{A}\,.\,d\,s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\lambda)$$

There will be the following equation similar to the last, due to V_1—

$$O = \frac{V_1}{2E}\int_A^B \frac{M\,x}{I}\,.\,d\,s + \frac{V_1}{2E}\int_A^B \frac{T\sin\phi}{A}\,.\,d\,s$$

$$+ \frac{V_1}{2G}\int_A^B \frac{S\cos\phi}{A}\,.\,d\,s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mu)$$

From equations (κ), (λ), (μ), H_1, V_1, and M_1 can be calculated, and all the other quantities, including the stresses, can be derived after these are known.

The Anchor Ring.

An anchor ring is an annulus, and its section is generally circular, though not necessarily so. It is required to find the resistance to rupture when strained in a manner similar to the links in a chain. A few links are shown in elevation on the left side of fig. 143. Consider any individual link. It is acted upon by two equal and opposite forces, each equal to the pull exerted by the neighbouring link. Let this pull be represented by W. Each link is symmetrical with respect to a vertical centre line, and consequently can be split up into two equal parts by that centre line, as shown in the middle of fig. 143. We shall then have $\frac{W}{2}$

acting, as shown, at each end of each half ring, and the action of W on the ring as a whole is to bend it to a smaller radius of curvature at the sections where the forces are applied. Let the moment due to this bending action be M_1; then at the top and bottom sections of the link one half of the link will act upon the other with a bending moment of M_1. On the right side of the figure is shown the centre line of a half link, acted upon by two equal and opposite forces each equal to $\frac{W}{2}$, and two equal and opposite moments

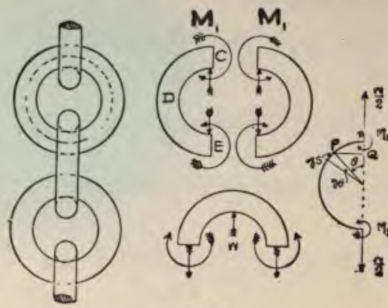

FIG. 143.

each equal to M_1. Consider any point P in this neutral line. The bending moment at P will be

$$\frac{W}{2} \cdot PQ - M_1 = \frac{W}{2} R \sin \theta - M_1,$$

where R is the radius of the neutral line. The change of angle between the ends of an element at P, whose length is $d\,s$, due to all the forces and couples, is*

$$d\,\phi = \frac{M}{EI} \cdot d\,s,$$

where ϕ is the angle between the ends of the element, and M is the bending moment over the element due to all the forces and couples,

$$= \frac{W}{2} R \sin \theta - M_1.$$

If the two halves of the link were hinged together at the top and bottom, there would be no bending moment M_1, and

* See also Appendix.

hence M_1 can be looked upon as the moment due to a redundant member, and the work done on the structure by M_1 must be zero.

Work done by M_1 in bending element through the change of angle $d\phi$

$$= M_1 \frac{d\phi}{2} = M_1 \frac{M\,ds}{2\,E\,I}.$$

The work done over the arc C D = work done over the arc D E, and work done over the whole ring = twice the work over C E = four times the work over the arc C D. Therefore the work done by M_1 in deforming structure

$$= o = 4\int_C^D M_1 \frac{M\,ds}{2\,E\,I} = \frac{2\,M_1}{E\,I}\int_C^D M\,ds = \frac{2\,M_1}{E\,I}\int_0^{\frac{\pi}{2}} M\,R\,d\theta.$$

or
$$o = \int_0^{\frac{\pi}{2}} M \,.\, d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{W}{2} R \sin\theta - M_1\right) d\theta$$

$$= \left[-\frac{W}{2} R \cos\theta - M_1\,\theta\right]_0^{\frac{\pi}{2}} = \tfrac{1}{2}(W\,R - \pi\,M_1)$$

or
$$M_1 = \frac{W\,R}{\pi}.$$

This result might have been obtained from slightly different considerations, thus : The total change of angle between the ends of the quadrant C D is zero, the sections at the ends remaining parallel to themselves ; hence by equation on page 298

$$o = \int_0^{\frac{\pi}{2}} d\phi = \int_0^{\frac{\pi}{2}} \frac{M\,ds}{E\,I} = \frac{R}{E\,I}\int_0^{\frac{\pi}{2}} \left(\frac{W}{2} R \sin\theta - M_1\right) d\theta$$

which gives the same result as above.

Let r be the radius of the section of the material of the ring, and f_b the stress produced in the outer fibres by bending. At D

$$M = \frac{W}{2} R - M_1$$

and $$f_b = \frac{M}{I}.r = \frac{r}{I}\left(\frac{WR}{2} - \frac{WR}{\pi}\right) = \frac{WRr}{\frac{1}{4}\pi r^4}\left(\tfrac{1}{2} - \frac{1}{\pi}\right)$$

$$= \frac{\cdot 235\ WR}{r^3}$$

At the same section

$$f_t = \frac{\frac{1}{2}W}{\pi r^2} = \cdot 159\frac{W}{r^2};$$

then

$$f_{max} = f_b + f_t = \frac{\cdot 235\ W\ R}{r^3} + \frac{\cdot 159\ W}{r^2}$$

$$= \frac{W}{r^3}(\cdot 235\ R + \cdot 159\ r).$$

Stress in Spokes of Bicycle Wheel.

First assume that the segments of the rim of the wheel are so rigid as not to alter their shape when a load is applied at the centre of the wheel. Let the load be W; then there will also be the pressure of the earth on the tyre upwards, equal to W. Now, from the equation,

$$n = 2j - 3,$$

it will be seen that there is one superfluous member, and that one is C L, in fig. 142. Only eight spokes are shown, but the method is the same, however many there are. Next assume the spokes are put in without any initial tension; then the load W will compress C L. But the spoke of a bicycle wheel is not capable of sustaining a compressive stress of any appreciable magnitude, and therefore we may conclude that C L in the position shown does not contribute to the stability of the wheel, and may therefore be removed. The line of action of the load W is produced through the centre of the wheel, and is represented by C L; while the reaction W, and the two segments C B and L B, are replaced by the two stresses C B and L B. We now have acting on the structure three external forces, namely, C L, C B, and L B. The stress diagram is found immediately underneath the wheel, and it will be noticed that the stresses in all the spokes are the same, while the stresses in the remaining segments are the same. This may have been found analytically by the method of sections, for cutting any pair of segments of the wheel with a line $p\,q$; and taking moments

about the centre, the stress in each segment must be the same, because the distance from the centre is constant. If the stress in each segment is the same, then the stress in each spoke must be the same.

Bicycle wheels are constructed so that the spokes have an initial tension before any load W is put on the wheel. Let T be this initial tension in each spoke; then, as the load is applied at the centre, the centre will be depressed, and all spokes above the horizontal through the centre will be elongated, while all the others will be shortened. Let v be this vertical deflection of the centre; then the work done on the wheel by the load W will be

$$\frac{W}{2} \times v.$$

This is spent in deforming the spokes of the wheel. Consider the action on the rth spoke, fig. 142, making an angle θ with spoke number 1. The rth spoke is elongated by the amount

$$d\,l_r = v \cos \theta.$$

If there are n spokes in the wheel, the circle will be divided by them into n equal angles, each of which is

$$\frac{2\,\pi}{n} \text{ radians,}$$

and consequently

$$\theta = \frac{2\,\pi}{n} \times r \text{ radians.}$$

Let S_r denote the increase of the stress above T in the rth spoke due to the load W applied at the centre; then

Initial stress before elongation $= T$.

Final stress after elongation $= T + S_r$.

Average stress during elongation $= \dfrac{2\,T + S_r}{2} = T + \dfrac{S_r}{2}$.

Work done during elongation $= \left(T + \dfrac{S_r}{2}\right) d\,l_r$.

Work done on all the spokes during application of load

$$= \left(T + \frac{S_1}{2}\right) d\,l_1 + \left(T + \frac{S_2}{2}\right) d\,l_2 + + +$$
$$+ \left(T + \frac{S_n}{2}\right) d\,l_n = \frac{W\,v}{2}$$

which may be simplified thus—

$$\frac{W\,v}{2} = T\,(d\,l_1 + d\,l_2 + d\,l_3 + + + d\,l_n)$$
$$+ \tfrac{1}{2}\,(S_1\,d\,l_1 + S_2\,d\,l_2 + S_3\,d\,l_3 + + + S_n\,d\,l_n).$$

Now, every spoke that is *elongated* will have a companion spoke in the same straight line that has been shortened a like amount, and consequently

$$d\,l_1 + d\,l_2 + d\,l_3 + + + d\,l_n = o\,;$$

hence

$$W\,v = S_1\,d\,l_1 + S_2\,d\,l_2 + S_3\,d\,l_3 + + + S_n\,d\,l_n.$$

It was shown in equation (δ) that

$$d\,l = S\,m, \text{ and therefore } S = \frac{d\,l}{m}\,;$$

hence

$$W\,v = \frac{d\,l_1{}^2}{m_1} + \frac{d\,l_2{}^2}{m_2} + \frac{d\,l_3{}^2}{m_3} + + + \frac{d\,l_n{}^2}{m_n}.$$

The spokes being all the same size, $m_1 = m_2 = m_3$, &c. Putting in the value of

$$d\,l_r = v \cos \frac{2\,\pi\,r}{n}$$

for each spoke we get—

$$W = \frac{v}{m}\left[\cos^2 \frac{2\,\pi}{n} + \cos^2 \frac{4\,\pi}{n} + \cos^2 \frac{6\,\pi}{n} + + + \cos^2 \frac{2\,n\,\pi}{n}\right]$$

Replace $\cos^2\theta$ by its equivalent,

$$\frac{1 + \cos 2\,\theta}{2},$$

and we have—

$$W = \frac{v}{m}\left[\frac{n}{2} + \frac{1}{2}\left\{\cos \frac{4\,\pi}{n} + \cos \frac{8\,\pi}{n} + \cos \frac{12\,\pi}{n} + + \cos \frac{4\,n\,\pi}{n}\right\}\right]$$

The series enclosed within the wave bracket is of the form—

$$\cos a + \cos 2\,a + \cos 3\,a + \cos 4\,a + + + \cos n\,a,$$

the sum of which is*

$$\frac{\cos\left(\frac{n+1}{2}\right) a \sin \frac{n\,d}{2}}{\sin \frac{a}{2}}.$$

* *Vide* Todhunter's "Plane Trigonometry," page 244, College edition.

Inserting this value in the above equation, we get

$$W = \frac{v}{2m}\left[n + \frac{\cos\left(\frac{n+1}{2}\right)\frac{4\pi}{n}\sin\frac{n}{2}\frac{4\pi}{n}}{\sin\frac{4\pi}{2n}}\right]$$

$$= \frac{vn}{2m},$$

because $\sin 2\pi$ is zero for all values of n;

also $$v = d\,l_1 = S_1\,m_1\,;$$

therefore $$W = \frac{S_1\,n}{2}.$$

Also $d\,l_2 = d\,l_1 \cos\frac{2\pi}{n} = v\cos\frac{2\pi}{n}$ and $d\,l_2 = S_2\,m_2$;

therefore $$W = \frac{vn}{2m} = \frac{S_2\,n}{2\cos\frac{2\pi}{n}},$$

and $$S_2 = \frac{2W}{n}\cos\frac{2\pi}{n};$$

also $$S_r = \frac{2W}{n}\cos\frac{2\pi}{n}(r-1),$$

and actual stress in spoke $= T + S_r$

$$= T + \frac{2W}{n}\cos\frac{2\pi}{n}(r-1).$$

If every spoke is in tension, and the stresses are a minimum, the least stress in the lowest spoke is zero, and

$$T = \frac{2W}{n}.$$

At the same time the stress in the highest spoke will be

$$T + \frac{2W}{n} = \frac{4W}{n}.$$

The segments of the rim have been assumed to be rigid; hence the effect is the same whether a load be applied at a single point in the rim or distributed symmetrically over it. Let the arrangement, fig. 142, be turned upside down, and we have the large Graydon wheel at Earls Court. Therefore, in that case, the uppermost vertical spoke has

no stress in it, while the lowest spoke has a stress equal to four times the load applied at its extremity. This stress is double the initial tension in the spoke.

Graphical Determination of Area of Plane Figure.*

Let it be required to find the area of the irregular plane figure, fig. 144, enclosed by the heavy line 7, *s*, *t*, &c. Select the point which occupies the extreme left of the figure. This point is 7. Through it draw a horizontal line extending on the right beyond the area to be measured. Divide the area above the horizontal through 7 into four-sided parts by vertical lines (shown dotted) through *s*, *t*, &c., and label the

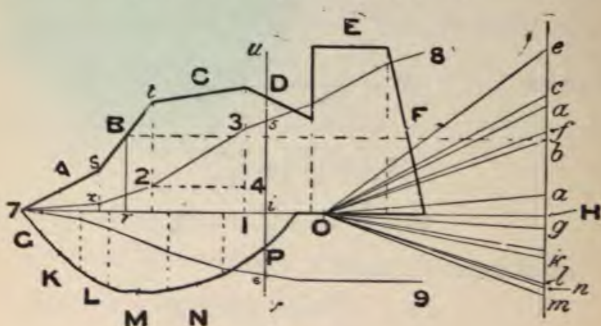

Fig. 144.

spaces between these vertical lines A, B, C, &c. At some point H in the horizontal through 7 draw the vertical line *e m*; then bisect each of the portions 7 *s*, *s t*, *t u*, &c., of the boundary line, and through those points of bisection draw horizontal lines to cut *e m* in *a*, *b*, *c*, &c. (one of these is shown through *b*). Now take any point O as pole in 7 H, and draw the radial lines shown. Beginning at the point 7, draw through the spaces A, B, C, &c., lines parallel to O *a*, O *b*, O *c*, &c., as in a funicular polygon. Then the area enclosed between the irregular outline, the vertical *u v*, and the horizontal 7 H, equals the true length of the intercept 5 *i* × true length of O H = 5 *i* × O H × scale of 5 *i* × scale of O H. This result can be proved thus : The area included between the verticals through *t* and 3 above the horizontal

* See also Appendix.

7 H = average height × width = true length of c H × true length of 2.4. = c H × 2.4. × scales. Also the triangle 2.3.4 is similar to the triangle O c H; therefore

$$\frac{3.4}{c\,\mathrm{H}} = \frac{2.4}{\mathrm{O\,H}} \text{ or } 3{\cdot}4 \times \mathrm{O\,H} = 2{\cdot}4 \times c\,\mathrm{H}.$$

But 2.4 × c H = area included between the two verticals through t and 3, if the figure is drawn full size; but if drawn to some scale, then the left-hand side of the equation must be multiplied by the scales to give the true area.

Now, the portion 3.4 of the intercept 3.1 is the vertical projection of 2.3, and consequently the area between the verticals through s and t will be given by the vertical projection of x.2 × O H × scales; also the area between 7 and the vertical through s will be given by the vertical projection of 7 x × O H × scales. Therefore the whole area above the horizontal 7 H between 7 and the vertical through 3 will be—(vertical projection of 7 x + vertical projection of x 2 + vertical projection of 2.3) × O H × scales = 3.1 × O H × scales. In the same way, the area up to the vertical $u\,i$ = 5 i × O H × scales. The area below the line 7 H can be represented in the same way, and consequently the area of the whole figure to the left of any vertical $u\,v$ = 5.6 × O H × scale of 5.6 × scale of O H.

THE END.

APPENDIX.

NOTE.—*The paragraphs in the Appendix will be preceded by the number of the pages in the book to which they refer; and generally the foot notes on the latter will indicate when additional matter has been placed in the Appendix.*

The Funicular Polygon (page 24). The proof of the funicular polygon method of determining bending moment, can be briefly stated as follows :—

Referring to figs. 12A and 12B on this page.

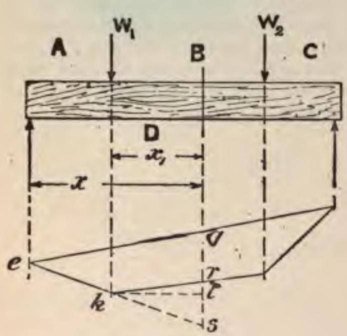

FIG. 12A.
STRUCTURE DIAGRAM.

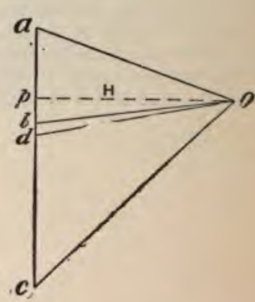

FIG. 12B.
FORCE DIAGRAM.

Take any section, say in the space B made by the line $s\ q$. distant x_1 from W_1 and x from A D, the left supporting force. The sides of the triangle $k\ r\ s$ (fig. 12A) have been drawn parallel to the sides $a\ o$, $o\ b$, and $b\ a$ (fig. 12B); hence these triangles are similar, and their sides proportional. Draw the horizontals $p\ o$ and $k\ t$. Then

$$\frac{a\ b}{s\ r} = \frac{b\ o}{k\ r} = \frac{o\ p}{k\ t}$$

But $ab = W_1$, and $k\,t = x_1$. Let op be called H, then the last equation may be written

$$\frac{W_1}{s\,r} = \frac{H}{x_1}$$

$$\text{or } W_1\,x_1 = H \times s\,r.$$

But $W_1\,x_1$, is the moment of W_1, about the section in question, and $s\,r$ is the intercept cut off by the two lines of the funicular polygon, which meet in the line of action of W_1. For the same reason $q\,s \times H$ must be the moment of A D about the same section. The bending moment at that section is

$$A\,D \times x - W_1\,x_1.$$

Substituting $q\,s \times H$ for the first term, and $s\,r \times H$ for the second, we have

$$q\,s \times H - s\,r \times H = H\,(q\,s - s\,r)$$
$$= H \times r\,q.$$

= horizontal polar distance in force diagram × vertical intercept of funicular polygon, under the section in question.

NOTE.—*The intercept* r q *must be measured on the same scale as that to which fig. 12a is drawn (it will be a length in feet or inches), and* H *must be measured on the scale to which fig. 12b is drawn (it will be a force in lbs. or tons).*

Articulated Structures. The reader cannot too carefully bear in mind the conditions under which it is possible to draw a stress diagram. The following hints may be of service in drawing these diagrams for the first time.

(1). Under all possible conditions of loading and support, the structure as a whole, and the individual members of the structure, must, each of them, obey the two laws of equilibrium given near the bottom of page 12.

(2). By "articulated structure" we mean a structure made up of parts hinged together, no single part having more than two frictionless hinges or pin connections. Each part behaves as if its axis were straight, and the stress in that member or part (for the purpose of the stress diagram) must act along the axis of the part as proved on page 14. This is tantamount to saying that there is no bending action in any of the parts or members of an articulated structure, or, if there is a bending action, then for the purpose of drawing the stress diagram, it is

assumed not to exist, because the stress diagram only takes account of orces along the axis of each member.

(3). For the purpose of drawing a stress diagram, forces can only be applied to an "articulated structure" at its

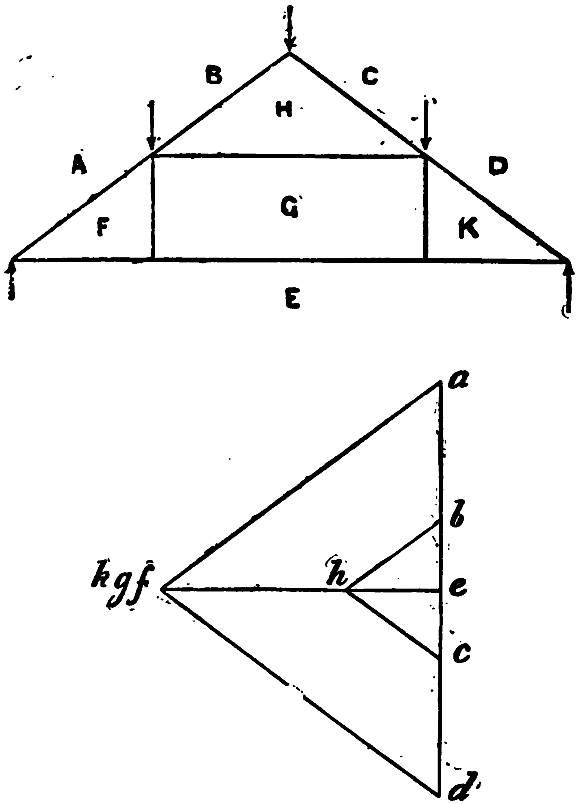

FIG. 17A.

joints or hinges; and should any member of a structure be called upon to resist a bending action, or if loads are applied at points other than the hinges or pin joints, then those loads must be divided up between the two

hinges of the member, in the same way that the load on a beam is divided into the two supporting forces. The stress diagram can then be drawn with the individual loads split up into equivalent loads at the joints or hinges.

(4). From paragraphs 1, 2, and 3, it is evident that the method of drawing the stress diagram of an "*articulated structure*" cannot be applied to the ordinary timber king-post truss, queen-post truss, or in fact any truss constructed with timber in the ordinary way; because the structure is not hinged, and it is purposely arranged with two or more members in one continuous piece. Again, in structures such as the queen-post truss, the essence of its stability is this continuation of some of its members, so that they contain more than *two* connecting points. The same applies to metal trusses, but rather less in degree, because the trusses are generally much

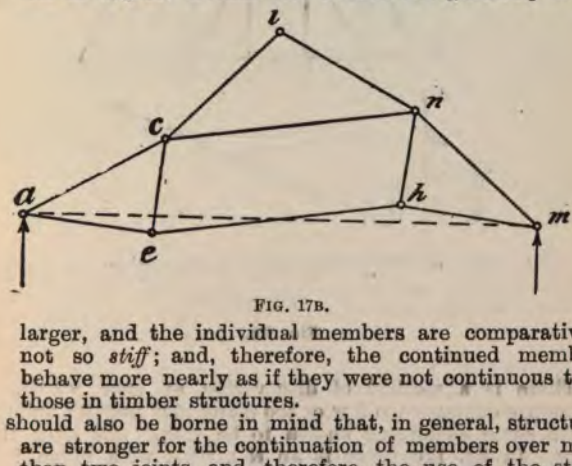

FIG. 17B.

larger, and the individual members are comparatively not so *stiff*; and, therefore, the continued members behave more nearly as if they were not continuous than those in timber structures.

It should also be borne in mind that, in general, structures are stronger for the continuation of members over more than two joints, and, therefore, the use of the stress diagram is quite safe.

The Queen Post Truss (fig. 17A) is a peculiar instance of an incomplete structure, *if it is assumed that its members are hinged as in an articulated structure*. This truss is shown in fig. 17A, and the stress diagram immediately under the truss. It will be noticed that F G and G K are not stressed.

Now the truss is in a state of unstable equilibrium, whether the members F G and G K are retained or not; for it is possible for it to assume the shape shown in fig. 17B, when it will collapse; in fact, in any other shape except that shown in fig. 17A the frame cannot exist without collapsing. The collapse may be prevented by making $a\,c$ and $c\,i$, fig. 17B, into one continuous member, as is done in practice. The members $i\,n$ and $n\,m$ are continuous, and $a\,e$, $e\,h$, and $h\,m$ are continuous, forming a stiff beam.

Another method of preventing distortion while the joints are still retained would be to prevent the diagonal $e\,n$ from lengthening, or $c\,h$ from shortening, by introducing a diagonal tie or strut.

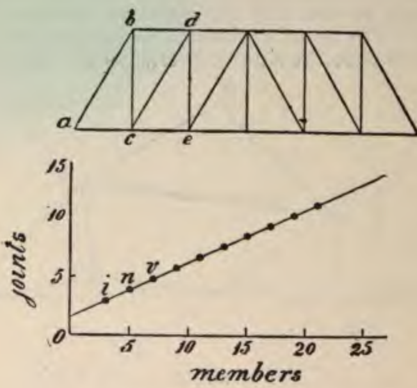

FIG. 17C.

There is a method of discovering at once whether a structure contains too few or too many members for stability. Take some structures known to be stable, and treat them thus. In fig. 17C there is an N girder, with hinge joints loaded at the joints of the bottom flange. Build up the structure from one end, taking care to note the number of members and joints at each stage. Thus, $a\,b\,c$ contains 3 members and 3 joints. Plot these numbers, as in the lower part of the figure at i. Add on the members $b\,d$ and $c\,d$, and we have 5 members and 4 joints; these are plotted at n. Next add the members $c\,e$ and $d\,e$, and then we have 7 members and 5 joints; these are plotted at v. Continuing

until the structure is complete, we have the points shown all situated on a straight line, whose equation is

Number of joints $= 1\frac{1}{2} +$ half the number of members.

or

$$\text{members} = 2 \times \text{joints} - 3.$$

If a structure has less than this number of members, it is not stable; while if it has more, some of them are superfluous.

Young's Modulus (page 45). The values given on page 44 must be looked upon as *low* average values, such as would be used in formulæ in design. Higher values are often obtained, but it is the function of the designer to make provision for variation in the quality of material, and hence the low values used. The amount of variation may be gathered from Unwin's Machine Design or Goodman's Applied Mechanics.

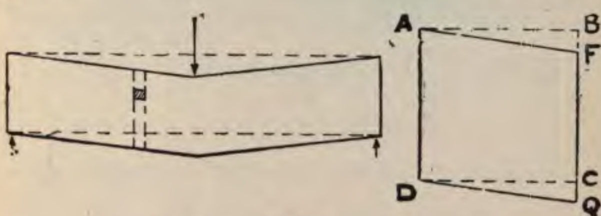

FIG. 20A.

Theory of Bending (page 45). It is assumed that when a beam is bent, the faces of the slice remain practically plane after bending. This is not strictly true, but the error is inappreciable in most cases (see below).

Table of Deflections (page 50). The quantities in the last column of this table represent the maximum deflection of a beam due to *bending*. There will, in addition, be a deflection due to *shearing*, but this is often very small compared with that due to bending. In the case of built-up plate girders, it may be very considerable. It may be found in the following manner:

In fig. 20A, a beam is shown, supported at the ends and loaded in the centre. Each small element, such as that shown shaded in fig. 20A, will be distorted from the original shape A B C D to A F Q D, by pure shear. Let the width of the beam be b, and f_s the shearing stress over the opposite

faces P Q and A D. The total shearing force over the face of an element such as $PQ = f_s \times$ area of face $= f_s \times FQ \times b$. The work done by this force in distorting the element is

$$f_s \times FQ \times b \times \frac{BF}{2}$$

But by definition, $\frac{BF}{AB}$ = shear strain,

$$\text{and } C = \frac{\text{shear stress}}{\text{shear strain}} = \frac{f_s}{\frac{BF}{AB}}$$

$$\text{or } BF = AB \times \frac{f_s}{C}$$

$$\text{and work done in deforming element} = f_s \times FQ \times b \times \frac{AB}{2} \times \frac{f_s}{C}$$

$$\text{hence work done per unit of volume of element} = \frac{f_s^{\,2} \times \frac{FQ \times b \times AB}{2C}}{AB \times BC \times b}$$

$$= \frac{f_s^{\,2}}{2C}$$

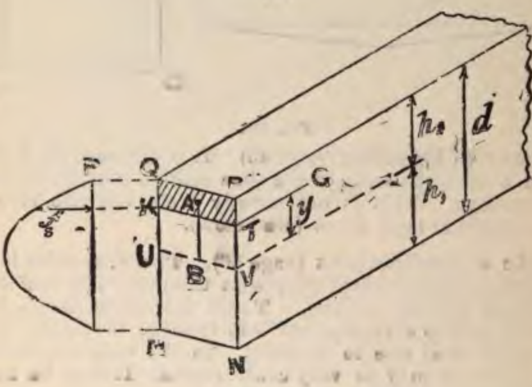

FIG. 20B.

It is shown on page 218 that the longitudinal shearing stress over a plane such as G T K (fig. 20B) parallel to the neutral surface was

$$f_s = \frac{F\,m}{b\,I}$$

Where F = shearing force over whole of transverse section Q P N M, b = breadth of section K T, I = geometrical moment of inertia of cross section about neutral line U V, and m = moment of the area Q P T K about the line U V.

This expression indicates that the shearing stress at any point in a beam increases from zero at P, to a maximum at the neutral surface V, as shown on the left side of fig. 20B.

As the shearing stress varies over a cross section, the work done in deforming a slice of a beam, such as that shown dotted in fig. 20A, will be the sum of the quantities of work done on the elements of which the slice is made up. The work done on the element shown shaded, fig. 20A,

$$= \frac{f_s^{\,2}}{2\,C} \times \text{volume of element} = \frac{f_s^{\,2}}{2\,C} \times d\,x.b.d\,y$$

Where $d\,x$ = length of slice, y = the mean distance of the element from the neutral surface, and f_s is the mean shear stress over the faces of the shaded element.

The work done in deforming the slice will then be

$$\int \left\{\frac{f_s^{\,2}}{2\,C} \times \text{volume of element}\right\} = \int \frac{f_s^{\,2}}{2\,C}\, b.\,d\,y.\,d\,x$$

$$= \frac{d\,x}{2\,C}\int b\,f_s^{\,2}\,d\,y$$

the integration being performed over the whole depth of the beam from $-h_1$ to $+h_2$.

The work done in deforming a piece of beam of length x, over which the shearing force is constant, must then be

$$\frac{x}{2\,C}\int_{-h_1}^{h^2} J_s^2 .\,dy$$

the equivalent f_s being substituted from

$$f_s = \frac{F\,m}{b\,I}$$

before the integration is performed.

If the beam is rectangular and uniform in section,

$$I = \frac{1}{12}\,b\,d^3 = \frac{2}{3}\,b\,h^3$$

and from fig. 20B,

$$m = PT \times b \times AB$$

$$= (h - y)\,b \times \left[y + \frac{h - y}{2}\right]$$

$$= (h - y)\frac{b}{2}(h + y)$$

$$= (h^2 - y^2)\frac{b}{2}$$

$$\text{and } f_s = \frac{F(h^2 - y^2)\frac{b}{2}}{b \times \frac{2}{3} b h^3}$$

$$= \frac{3}{4}\frac{F}{b h^3}(h^2 - y^2)$$

Inserting this in the expression for work done, we get the work done over length x

$$= \frac{x b}{2 C}\int_{-h}^{h}\left[\frac{3}{4}\frac{F}{b h^3}(h^2 - y^2)\right]^2 dy$$

$$= \frac{9}{32 b}\frac{F^2 x}{h^6 C}\int_{-h}^{h}(h^4 - 2 h^2 y^2 + y^4)\, dy$$

$$= \frac{3}{10} \cdot \frac{F^2 x}{b h C} \ *$$

Taking the case of a single load W at the centre of the beam, then $F = \frac{1}{2}$ W, and the work done is

$$\frac{3}{10} \times \frac{W^2}{4} \times \frac{x}{b h C}$$

Now the work done by *bending* a slice of the beam apart from shearing

$= \frac{1}{2}$ moment on slice $\times$ angle turned through by ends of slice.

$= \frac{1}{2}$ M. $d\phi$ for a length dx of beam.

But $d\phi$ is the corresponding increment of slope, which is

$$\frac{M}{E I} . dx.$$

Hence the work done in bending $= \int_0^x \frac{1}{2}\frac{M^2}{E I} . dx.$

* If the shearing force F is not constant over the length x, then the value of F must be inserted in terms of x, and the work done will then be

$$\frac{3}{10}\int_0^x \frac{F^2}{b h C} . dx$$

And with the central load W considered above, this expression becomes

$$\frac{W^2 X^3}{24 E I} \text{ or } \frac{W^2 X^3}{16 E b h^3}$$

Take l = half the length of beam, and d the deflection at the centre; then

$\frac{W}{2} \times d = 2 \times$ work due to *bending* half the beam $+ 2 \times$ work due to *shearing* half the beam.

$$= \left[\frac{W^2 l^3}{16 E b h^3} + \frac{3}{40} \frac{W^2 l}{b h C}\right] 2$$

$$\text{and } d = \frac{W l^3}{4 E b h^3} + \frac{3}{10} \frac{W l}{b h C}$$

The first term on the right is the deflection due to bending, and the other that due to shear; hence

$$\frac{\text{deflection due to shear}}{\text{deflection due to bending}} = \frac{\frac{3}{10} \frac{W l}{b h C}}{\frac{W l^3}{4 E b h^3}} = \frac{6}{5} \frac{h^2}{l^2} \frac{E}{C}$$

But $C = \frac{2}{5}$ E approximately, hence we have for a beam of rectangular section:*

deflection due to shear $= \frac{3 h^2}{l^2} \times$ deflection due to bending.

Assuming the ratio $\frac{h}{l} = \frac{1}{10}$, the ratio of the deflections is 1 to 33. Also when $\frac{h}{l} = \frac{1}{20}$, the ratio becomes 1 to 133.

When the shear stress is constant or approximately so, over a cross-section, as is the case with a plate girder or rolled joist, the above method will give the ratio of the deflections, but the calculation is much simplified on account of f_s being constant over the section.

$$\text{We shall then have} \quad f_s = \frac{F}{a}$$

Where F is the shearing force and a the area of cross-section of the web, which in this case resists the shear alone (see page 220).

* h is the half depth of a beam and l the half length.

Then work done n deforming slice = volume $\times \frac{f_s^2}{2\,C}$

$$= 2\,h.\,b.\,d\,x.\,\frac{f_s^2}{2\,C} = \frac{h.\,b.\,d\,x.}{C}\left(\frac{F}{a}\right)^2$$

$$= \frac{F^2.\,d\,x}{2\,C\,a}$$

And as F and a are independent of x, the work done in deforming a length x of the beam must be

$$\frac{F^2}{2\,C\,a}\int_0^x .\,d\,x = \frac{F^2\,x}{2\,C\,a}$$

With a single central load as in the case of the solid beam, $F = \frac{1}{2}\,W$, and the work done in deforming the whole beam is

$$\frac{W^2\,l}{4\,C\,a}$$

when a is the depth and b are constant, and l = half length. And by the previous reasoning, the deflection due to shearing alone must be

$$\frac{W\,l}{2\,C\,a} \text{ or } \frac{W\,L}{4\,C\,a}$$

Where L = the whole length of the beam, the deflection due to bending alone is

$$\frac{1}{48}\,\frac{W\,L^3}{E\,I} = \frac{W\,L^3}{48\,E\,2\,A\,h^2}$$

Where A is the sectional area of one flange, and the web is neglected, the depth, width, and cross-section being constant,

Then

$$\frac{\text{deflection due to shearing}}{\text{deflection due to bending}} = \frac{\frac{W\,L}{4\,C\,a}}{\frac{W\,L^3}{96\,E\,A\,h^2}}$$

$$= \frac{24\,E\,A\,h^2}{C\,a\,L^2} = \frac{60\,A\,h^2}{a\,L^2}$$

As an example, let the web be $\frac{3}{8}$ in. wide and 12 inches deep, and the area of one flange be 3 sq. ins., while h is $6\frac{1}{4}$ inches and L is 300 inches. Then the ratio of the deflections

$$= \frac{60 \times 3 \times 6{\cdot}25^2}{12 \times \frac{3}{8} \times 300^2} = \frac{1}{57}$$

If the length L had been 12·5 ft., the ratio would have been $\frac{1}{14}$. Had the beam been a built-up plate girder

with a web 20 × $\frac{3}{8}$ inches, and length 200 inches, while the area of one flange was 10 sq. inches, the ratio of deflections would have been $\frac{1}{5}$. Further, had the flanges been reduced in section to the amount required to resist the bending moment the deflection due to bending would have been :

$$\frac{W L^3}{10 A h^2}$$

and the ratio of the deflections $\frac{10 A h^2}{a L^2}$

The deeper the girder in proportion to its length, the more nearly does the deflection due to shear approach that due to bending.

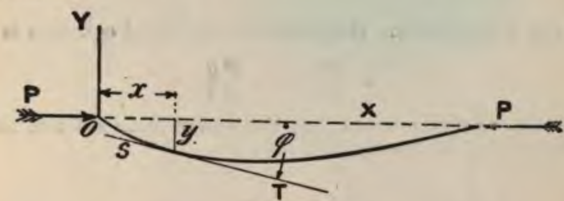

FIG. 35A.

The Crane Hook Problem (page 69). Professor Karl Pearson has recently shown that there may be considerable error in applying equation 57 to the crane hook or any other body whose axis is not *straight* at the section in question. The error comes about by using equation 29, which only refers to *straight* beams. With curved beams, the change of curvature must be taken into account, as it materially affects the result when the radius of curvature is not large.

The theory is complicated, but the results are important. In some cases, notably a drawbar hook, equation 57 indicated a stress of 12,000 lbs. per sq. in., while the real stress was 23,000 lbs. per sq. in. Readers should consult The Drapers' Company Research Memoir I., by E. S. Andrews and Professor Karl Pearson, published by Dulan & Co., 37, Soho Square, London, W.

The Sign of the Terms in the Equation (page 80).

$$\frac{d^2 y}{d x^2} = \frac{M}{E I}$$

In fig. 35A, y is measured in the negative direction from the axis of x, and hence

$$\frac{M}{EI} = \frac{P(-y)}{EI} = \frac{-Py}{EI}$$

As regards the left hand side, $\frac{d^2 y}{d x^2}$ is $\frac{d}{d x}\left(\frac{d y}{d x}\right) = \frac{d}{d x} . tan \phi$

Now in fig. 35A, the increase of $tan \phi$ is positive because the tangent S T rotates in the positive direction with an increase of x, which is what it would have to do, to increase a positive angle or decrease a negative angle. Or thus, in fig. 35A, $tan \phi$ is negative, and, as x increases, ϕ becomes less; hence $\frac{d}{d x} tan \phi$ is a negative increase of a negative quantity; *i.e.*, $\frac{d}{d x} tan \phi$ is positive. Hence the fundamental equation is

$$+ \frac{d^2 y}{d x^2} = - \frac{P y}{EI}$$

It must be remembered that the sign of both sides of this equation must be determined.

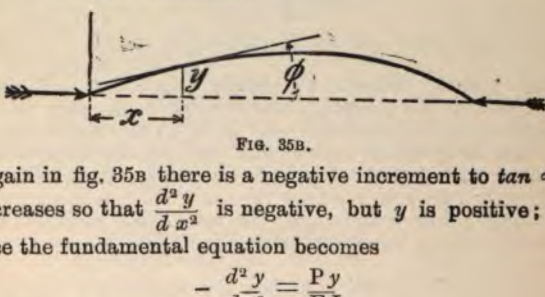

FIG. 35B.

Again in fig. 35B there is a negative increment to $tan\ \phi$, as x increases so that $\frac{d^2 y}{d x^2}$ is negative, but y is positive; and hence the fundamental equation becomes

$$- \frac{d^2 y}{d x^2} = \frac{P y}{EI}$$

which is the same as the one above.

Wind Pressure (page 140). The pressure of wind on a roof is given approximately by the following simple expression—

Normal pressure in lbs. per sq. ft.
$= 3 +$ slope of roof in degrees.

This is a much more convenient expression than that of Hutton on page 140.

French Truss (page 141). Another method of getting over the difficulty is as follows—

The left half of the principal is really an articulated girder, as shown in fig. 63A. It is supported by a vertical reaction V on the left, the horizontal reaction H of the right half on it and another horizontal force *h* in the tie rod across the centre.

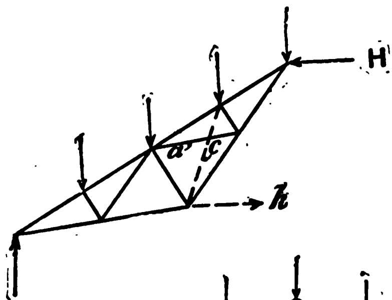

FIG. 63A.

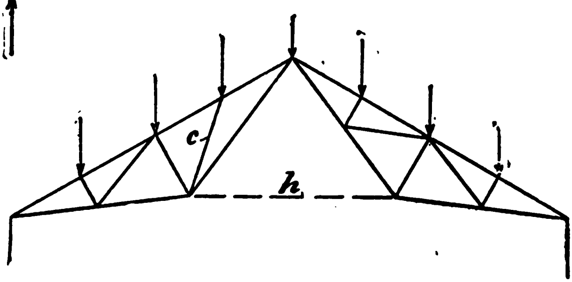

FIG. 63B.

Now, these supporting forces will be the same whatever the bracing of the girder, hence, change the direction of the brace *a* to that of *c* as shown dotted in fig. 63A, and full in fig. 63B. There will now be no difficulty in drawing the stress diagram until the stress *h* has been obtained, after which replace *c* by *a* and return to the original construction.

The Quadrangular Truss (chap. xviii., fig. 83) is really made up of double cantilevers supporting ordinary trusses. This may be more easily seen on reference to fig. 83A, where the cantilever has been separated from the ordinary truss, the dotted lines being put in for appearance. They also add to the stiffness of the structure. The cantilever is not really constructed as indicated in the figure where it appears to *rest*

upon the end of a pillar. The pillar is continued up to the gutter and the cantilever built to it. This also makes a much stiffer construction.

Counterbracing (chap. xxi.). A quicker method of determining the panels that require counterbracing is to proceed thus—fig 89A. The bracing in an articulated structure is for the purpose of resisting the shearing force.

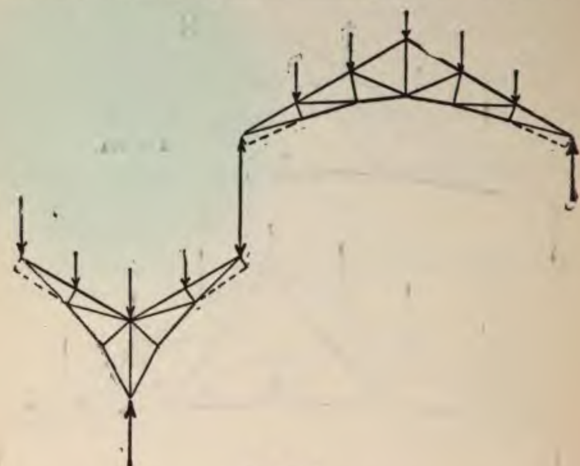

FIG 83A.

The shearing force diagram due to dead load only is shown at the top of fig. 89A, in which the ordinate plotted above $k p$ such as $h d$ represents the tendency of the part $d p$ to slide *upwards* relative to the part $k d$.

In the same way the diagram immediately below represents the maximum shearing force due to the moving load, when that load approaches from the left. Thus $c m$ represents the maximum shearing force at the point m due to the moving load only. As this is of the same sign as the dead load shearing force $h d$, they may be added as shown in the lowest part of the figure. Thus $e n = e a + a n = c m + h d$, giving the outline $u t e l b y u$. But the load may approach from the right, and then the combined shear force diagram is $u r i j d y u$.

Between *y* and *l* the shearing force is always negative, and between *j* and *u* it is always positive; but between *j* and *l* it is sometimes positive and sometimes negative, necessitating

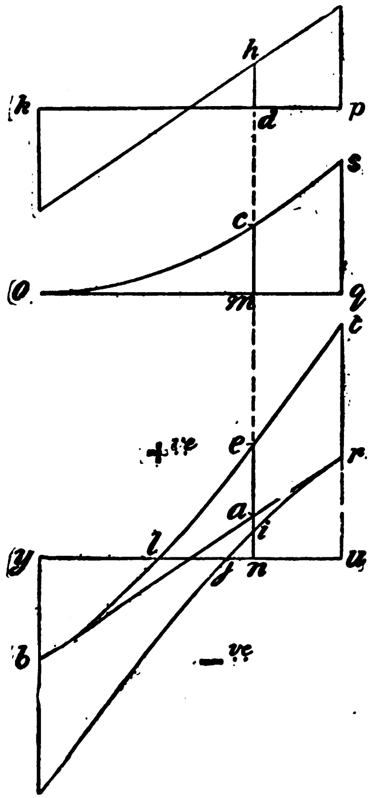

FIG. 89A.

the design of the bracing, so that it is capable of resisting positive *and* negative stress. This can be done, as explained in the text, by (1) making the ties capable of acting as struts, or (2) by putting in duplicate ties coincident with the opposite diagonal of each panel.

Moment of Inertia (chap. xxii.). A shorter method of graphically determining the moment of inertia, or second moment of area of an irregular figure, is as follows:—

In fig. 93A, $afmbk$ is an irregular area. The centre line ab of a horizontal strip is drawn, whose width is w. The area of this strip is $ab \times w$, and the moment of this area about the axis ph is $ab \times w \times ap. = w \times nq \times ap$. Now

$$\frac{ap}{np} = \frac{ac}{nq}$$

or

$$ap \times nq = ac \times np = ac \times d,$$

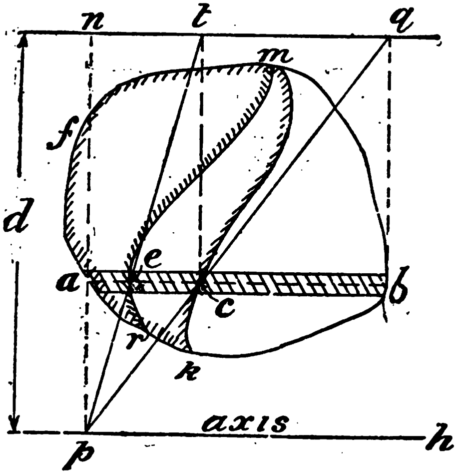

FIG. 93A.

and, therefore, the moment of the strip about the axis is

$$w \times nq \times ap = w \times ac \times d$$
$$= \text{area of strip } ac \times d,$$

and similarly the moment of the area of the whole figure about the axis must be:

$$\text{area } kafmck \times d.$$

This area is shown shaded inside the boundary.

The moment of inertia of the strip al about the axis ph is

$$\text{area } ab \times ap^2 = w \times ab \times ap^2$$
$$= w \times ab \times ap \times ap$$
$$= \text{area } ac \times d \times ap$$

and, by the same reasoning as given above, this must equal

$$d \times \text{area } ae \times d$$

$$\text{or } d^2 \times ae$$

Similarly, the moment of inertia of the whole figure about the axis must be

$$\text{area } rafmer \times d^2$$

Hence determine this area preferably with a planimeter, and multiply by d^2.

Masonry Structures (page 237). In the case of an annular section,

$$y = \frac{3}{8} D - \frac{d^2}{8 D}$$

Where D = external diameter of the annulus, and d = the internal diameter.

In the case of a hollow square, whose outer dimension is D, and inner dimension d,

$$y = \frac{D}{3} - \frac{d^2}{6 D}$$

It should be remembered that although no tension is assumed to exist in parts of masonry structures, yet it does exist, especially where cement has been used, and the structure is thereby considerably strengthened.

Stability of Masonry Dam (page 250). The method of ascertaining the stability described in the text is that usually adopted, but Professor Karl Pearson has recently shown* that it is insufficient, and that there is considerable tension in the masonry near the tail of the dam, and that the *vertical* sections are more dangerous than the horizontal sections. The original Memoir should be consulted for further information on the subject.

Fixed Arch Ring (page 308). Equation k is obtained as follows :—

$\frac{M}{E I}$ = change of curvature at a point in the neutral axis due to the bending moment M at that point.

Let R_0 = original radius of curvature,

and R = radius of curvature after the application of the moment M.

$$\text{Then change of curvature} = \frac{1}{R} - \frac{1}{R_0}$$

* Drapers' Company Research Memoirs II., on Masonry Dams, by L. W. Atcherley and Karl Pearson. Published by Dulan & Co., 37, Soho Square, London, W.

Let ϕ be the inclination of the tangent to the horizon, then $d\phi_0$ is the angle between the tangents at the ends of the element before bending, and $d\phi$ the same after bending. These will also be the angles between the radii of curvature at the ends of the element before and after bending, and hence

$$R_0 \,.\, d\phi_0 = ds$$

$$\text{and } \frac{1}{R_0} = \frac{d\phi_0}{ds}$$

$$\text{similarly } \frac{1}{R} = \frac{d\phi}{ds}$$

$$\text{and } \frac{1}{R} - \frac{1}{R_0} = \frac{d\phi - d\phi_0}{ds}$$

$= \dfrac{1}{ds} \times$ change in the angle between radii at ends of element due to moment M.

$$\text{but } \frac{M}{EI} = \frac{1}{R} - \frac{1}{R_0}$$

$$\text{hence, total change of angle} = \int \frac{M}{EI} \,.\, ds.$$

Area of Plane Figure (page 318). There is some very interesting matter connected with this work in Lineham's Text Book of Mechanical Engineering, page 851, where methods are given which are superior to that given on page 318.

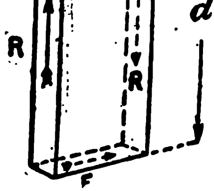

FIG. 103.

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